Pair-copula Constructions of Multiple Dependence

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Reference: Aas, Czado, Frigessi, and Bakken (2007)

Overview

Introduction

- Bedford and Cooke (2001) and Bedford and Cooke (2002) gave a probabilistic construction of multivariate distributions based on simple building blocks called pair-copulas. See also Kurowicka and Cooke (2006).
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- Extends work by Joe (1996).
- Overseen so far by statisticians and econometricians.
- We believe that they provide a central tool in model building, when conditional independence is not natural
- Maintains the logic of building complexity by simple elementary blocks.

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Multivariate modeling

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  - Not appropriate when all variables do not have the same distribution.
- Marginal distribution + copula
  - Not appropriate when all pairs of variables do not have the same dependency structure.
- Marginal distributions + pair-copula construction
  - Very flexible structure

Multivariate distributions

Consider n random variables $\mathbf{X} = (X_1, \ldots, X_n)$ with

- joint density $f(x_1, \ldots, x_n)$ and marginal densities $f_i(x_i), i = 1, \ldots, n$
- joint cdf $F(x_1, \ldots, x_n)$ and marginal cdf's $F_i(x_i), i = 1, \ldots, n$

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- joint cdf $F(x_1, \ldots, x_n)$ and marginal cdf's $F_i(x_i), i = 1, \ldots, n$
- factorization

$$f(x_1, \ldots, x_n) = \\ f_n(x_n) \cdot f(x_{n-1}|x_n) \cdot f(x_{n-2}|x_{n-1}, x_n) \ldots \cdot f(x_1|x_2, \ldots, x_n) \tag{1}$$

Copula

A copula is a multivariate distribution on $[0,1]^n$ with uniformly distributed marginals.

- copula cdf $C(u_1, \ldots, u_n)$
- copula density $c(u_1, \ldots, u_n)$

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Using Sklar's Theorem (1959) we have for absolutely continuous distributions with continuous marginal cdf's

$$f(x_1, \ldots, x_n) = c_{12\cdots n}(F_1(x_1), \ldots F_n(x_n)) \cdot f_1(x_1) \cdots f_n(x_n) \qquad (2)$$

for some n-variate copula density $c_{12\cdots n}(\cdot)$.

Pair-copula constructions (PCC)

Copula decompositions (1)

- $n = 2$

$$
\begin{aligned}
&\text{joint density} \\
f(x_1, x_2) &= c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2) \\
&\text{conditional densities} \\
f_{1|2}(x_1|x_2) &= c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \\
f_{2|1}(x_2|x_1) &= c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2)
\end{aligned}
$$

Copula decompositions (2)

- $n = 3$
  - first decomposition:

$$
\begin{aligned}
f_{1|23}(x_1|x_2, x_3) &= \frac{f_{12|3}(x_1, x_2|x_3)}{f_{2|3}(x_2|x_3)} \\
&= c_{12|3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3)) \cdot f_{1|3}(x_1|x_3)
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  - Second decomposition:

$$\begin{aligned} f(x_1|x_2,x_3) &= c_{13|2}(F_{1|2}(x_1|x_2),F_{3|2}(x_3|x_2)) \cdot f(x_1|x_2) \\ &= c_{13|2}(F_{1|2}(x_1|x_2),F_{3|2}(x_3|x_2)) \\ &\quad \times c_{12}(F_1(x_1),F_2(x_2)) \cdot f_1(x_1) \end{aligned}$$

Copula decompositions (3)

- general n
  - For $\mathbf{v} = (v_1, \ldots, v_d)$ and any $j = 1, \ldots, d$

$$
\begin{aligned}
f(x|\mathbf{v}) &= c_{xv_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j})) \cdot f(x|\mathbf{v}_{-j}) \\
\mathbf{v}_{-j} &= (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_d) \\
c_{xv_j|\mathbf{v}_{-j}}(\cdot) &= \text{bivariate copula density}
\end{aligned}
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  - Combining this decomposition of the conditional distribution with the factorization (1), we derive at a decomposition of $f(x_1, \dots, x_n)$ that only consist of pair-copulae.
  - We call this a pair-copula construction.

Conditional cdf's

- Univariate v: Since $f(x|v) = c_{xv}(F_x(x), F_v(v))f_x(x)$ we have

$$F(x|v) = \int_{-\infty}^{x} c_{xv}(F_x(u), F_v(v))f_x(u)du$$

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&= \frac{\partial}{\partial F_v(v)} \int_{-\infty}^{x} \frac{\partial C_{xv}(F_x(u), F_v(v))}{\partial F_x(u)} \frac{dF_x(u)}{du} du
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&= \frac{\partial}{\partial F_v(v)} \int_{-\infty}^{x} \frac{\partial C_{xv}(F_x(u), F_v(v))}{\partial F_x(u)} \frac{dF_x(u)}{du} du \\
&= \frac{\partial\, C_{xv}(F_x(x), F_v(v))}{\partial F_v(v)}
\end{aligned}
$$

Conditional cdf's (2)

- In the case of uniform marginal cdf's, i.e. $F_x(x) = x$ and $F_v(v) = v$ we have for parametrized $C_{xv}(x, v, \Theta)$

$$h(x, v, \Theta) := F(x|v) = \frac{\partial\, C_{xv}(x, v, \Theta)}{\partial v}.$$

These h functions will become essential for simulation and inference.

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$$h(x, v, \Theta) := F(x|v) = \frac{\partial\, C_{xv}(x, v, \Theta)}{\partial v}.$$

These h functions will become essential for simulation and inference.

- Under regularity conditions Joe (1996) showed that

$$F(x|\mathbf{v}) = \frac{\partial\, C_{x,v_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j}))}{\partial F(v_j|\mathbf{v}_{-j})},$$

Examples of PCC's

- $n = 3$

$$f(x_1, x_2, x_3) = f_3(x_3) f(x_2|x_3) f(x_1|x_2, x_3)$$

Examples of PCC's

- $n = 3$

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\begin{aligned}
f(x_1, x_2, x_3) = & \\
& f_3(x_3) f(x_2|x_3) f(x_1|x_2, x_3) \\
= \quad & f_3(x_3)[c_{23}(F_2(x_2), F_3(x_3)) f_2(x_2)] \\
& \times [c_{12|3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3)) f(x_1|x_3)]
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= & f_3(x_3)[c_{23}(F_2(x_2), F_3(x_3)) f_2(x_2)] \\
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& \times [c_{13}(F_1(x_1), F_3(x_3)) f_1(x_1)] \\
= \; & f_1(x_1) f_2(x_2) f_3(x_3) \times c_{23}(F_2(x_2), F_3(x_3)) \\
& \times c_{13}(F_1(x_1), F_3(x_3)) c_{12|3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3))
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= & \ f_3(x_3)[c_{23}(F_2(x_2), F_3(x_3)) f_2(x_2)] \\
& \times [c_{12|3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3)) f(x_1|x_3)] \\
= & \ f_3(x_3)[c_{23}(F_2(x_2), F_3(x_3)) f_2(x_2)] \\
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& \times [c_{13}(F_1(x_1), F_3(x_3)) f_1(x_1)] \\
= & \ f_1(x_1) f_2(x_2) f_3(x_3) \times c_{23}(F_2(x_2), F_3(x_3)) \\
& \times c_{13}(F_1(x_1), F_3(x_3)) c_{12|3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3))
\end{aligned}
$$

If X_1 ind. X_2 given X_3 $\Rightarrow c_{12|3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3)) = 1$

- $n = 5$

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4, x_5) = \\
& f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \cdot f_4(x_4) \cdot f_5(x_5) \\
\cdot \quad & c_{12}\left(F_1(x_1), F_2(x_2)\right) \cdot c_{13}\left(F_1(x_1), F_3(x_3)\right) \cdot c_{14}\left(F_1(x_1), F_4(x_4)\right) \\
\cdot \quad & c_{15}\left(F_1(x_1), F_5(x_5)\right) \cdot c_{23|1}\left(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1)\right) \\
\cdot \quad & c_{24|1}\left(F_{2|1}(x_2|x_1), F_{4|1}(x_4|x_1)\right) \cdot c_{25|1}\left(F_{2|1}(x_2|x_1), F_{5|1}(x_5|x_1)\right) \\
\cdot \quad & c_{34|12}\left(F_{3|12}(x_3|x_1, x_2), F_{4|12}(x_4|x_1, x_2)\right) \\
\cdot \quad & c_{35|12}\left(F_{3|12}(x_3|x_1, x_2), F_{5|12}(x_5|x_1, x_2)\right) \\
\cdot \quad & c_{45|123}\left(F_{4|123}(x_4|x_1, x_2, x_3), F_{5|123}(x_5|x_1, x_2, x_3)\right),
\end{aligned}
$$

There are 240 different such constructions for $n = 5$

Vines

- Hence, for high-dimensional distributions there are many possible pair-copula constructions.
- Bedford and Cooke (2001) introduced a graphical model called regular vine to help organize them.
- The class of regular vines is large and embraces a large number of possible PCC's.

Vines

- Hence, for high-dimensional distributions there are many possible pair-copula constructions.
- Bedford and Cooke (2001) introduced a graphical model called regular vine to help organize them.
- The class of regular vines is large and embraces a large number of possible PCC's.
- We concentrate on two special cases (Kurowicka and Cooke 2004):
  - D-vine
  - Canonical Vine

Vines(2)

- An *n*-dimensional vine is represented by n-1 trees.

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- Each edge corresponds to a pair-copula density.
- Edges in tree j become nodes in tree $j+1$.
- Two nodes in tree $j+1$ are joined by an edge if the corresponding edges in tree j share a node.
- The complete decomposition is defined by the $\frac{n(n-1)}{2}$ edges (i.e. pair copula densities) and the marginal densities.

Canonical and D-vines

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  A regular vine for which each tree has a unique node that is connected to $n - j$ edges.

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- D-vine:
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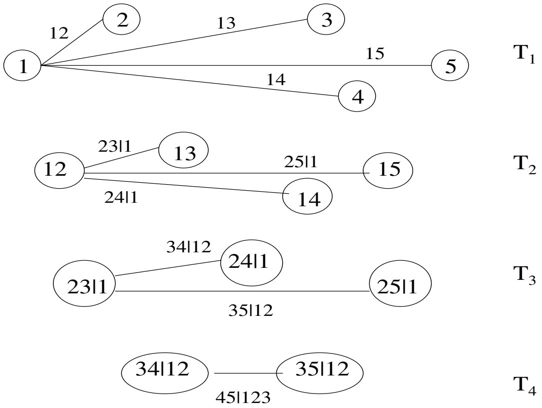
- Abbreviations:

$$
\begin{aligned}
c_{ij|\mathbf{v}} &:= c_{ij|\mathbf{v}}(F_{i|\mathbf{v}}(x_i|\mathbf{x_v}), F_{j|\mathbf{v}}(x_j|\mathbf{x_v})) \\
f_j &:= f_j(x_j) \\
f_{\mathbf{v}} &:= f_{\mathbf{v}}(x_{\mathbf{v}})
\end{aligned}
$$

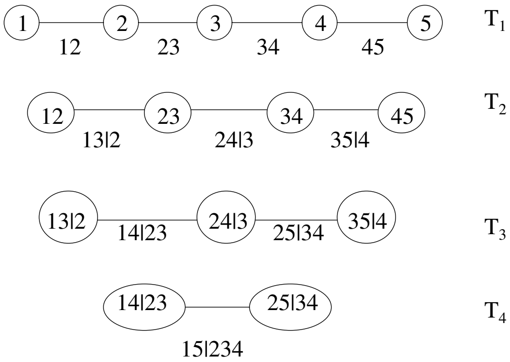
Five dimensional canonical vine

$$f_{12345} = f_1 \cdot f_2 \cdot f_3 \cdot f_4 \cdot f_5 \cdot c_{12} \cdot c_{13} \cdot c_{14} \cdot c_{15} \cdot c_{23|1} \cdot c_{24|1} \cdot c_{25|1} \cdot c_{34|12} \cdot c_{35|12} \cdot c_{45|123}$$



Five dimensional D-vine

$$f_{12345} = f_1 \cdot f_2 \cdot f_3 \cdot f_4 \cdot f_5 \cdot c_{12} \cdot c_{23} \cdot c_{34} \cdot c_{45} \cdot c_{13|2} \cdot c_{24|3} \cdot c_{35|4} \cdot c_{14|23} \cdot c_{25|34} \cdot c_{15|234}$$



General density expressions

- Canonical vine density

$$\prod_{k=1}^{n} f_k \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{j,j+i|1,\dots,j-1}$$

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- Canonical vine density

$$\prod_{k=1}^{n} f_k \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{j,j+i|1,\ldots,j-1}$$

- D-vine density

$$\prod_{k=1}^{n} f_k \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{i,i+j|i+1,\ldots,i+j-1}$$

where index j identifies the trees, while i runs over the edges in each tree.

Simulation of Pair-copula constructions

For simplicity we assume that the marginal distributions are uniform, i.e. $f_k(x_k) = 1$ and $F_k(x_k) = x_k$ for $k = 1, \ldots, n$. Both canonical and D-vines are simulated as follows:

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- Sample $w_i, i = 1, \ldots, n$ independent uniform on [0,1].
- Set

$$
\begin{aligned}
x_1 &= w_1 \\
x_2 &= F^{-1}_{2|1}(w_2|x_1) \\
x_3 &= F^{-1}_{3|1,2}(w_3|x_1, x_2) \\
.. &= .... \\
x_n &= F^{-1}_{n|1,2,\ldots,n-1}(w_n|x_1, \ldots, x_{n-1}).
\end{aligned}
$$

The simulation procedure for the canonical and D-vine differs in how $F(x_j|x_1, x_2, \ldots, x_{j-1})$ is computed.

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- For the canonical vine $F(x_j|x_1, x_2, \ldots, x_{j-1})$ is computed as

$$\frac{\partial\, C_{j,j-1|1,\ldots j-2}(F(x_j|x_1, \ldots, x_{j-2}), F(x_{j-1}|x_1, \ldots, x_{j-2}))}{\partial F(x_{j-1}|x_1, \ldots, x_{j-2})}.$$

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- For the D-vine $F(x_j|x_1, x_2, \ldots, x_{j-1})$ is computed as

$$\frac{\partial\, C_{j,1|2,\ldots j-1}(F(x_j|x_2, \ldots, x_{j-1}), F(x_1|x_2, \ldots, x_{j-1}))}{\partial F(x_1|x_2, \ldots, x_{j-1})}.$$

Simulation for $n = 3$

For $n = 3$ we have canonical = D-vine

- Sample $w_i, i = 1, \ldots, 3$ i.i.d $U[0, 1]$
- Set $x_1 = w_1$

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- Sample $w_i, i = 1, \ldots, 3$ i.i.d $U[0, 1]$
- Set $x_1 = w_1$
- Set $x_2 = h^{-1}(w_2, x_1, \Theta_{12})$, since $F_{2|1}(x_2|x_1) = h(x_2, x_1, \Theta_{12})$

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- Set $x_1 = w_1$
- Set $x_2 = h^{-1}(w_2, x_1, \Theta_{12})$, since $F_{2|1}(x_2|x_1) = h(x_2, x_1, \Theta_{12})$
- $x_3 = h^{-1}(h^{-1}(w_3, h(x_2, x_1, \Theta_{12}), \Theta_{23|1}), x_1, \Theta_{13})$ since

$$
\begin{aligned}
F(x_3|x_1, x_2) &= h(F_{3|1}(x_3|x_1), F_{2|1}(x_2|x_1), \Theta_{23|1}) \\
&= h(h(x_3, x_1, \Theta_{13}), h(x_2, x_1, \Theta_{12})), \Theta_{23|1})
\end{aligned}
$$

Simulation algorithms for general n can be found in Aas, Czado, Frigessi, and Bakken (2007).

Model selection

Inference for a pair-copula constructions should in principle consider

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3. The estimation of the parameters of the chosen pair-copulae

Number of PCC's and some smart choices

- There are $\frac{n!}{2}$ different canonical vines and $\frac{n!}{2}$ different D-vines
- For $n > 3$ canonical and D-vines are always different
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- For $n > 3$ canonical and D-vines are always different
- There are many regular vine decompositions that are neither canonical nor D-vines
- For $n \leq 4$ one can estimate parameters for all decompositions if only one type of pair-copulae is used.
- For higher dimensions, one can first estimate most important bivariate relationships and use them to determine which decomposition to use.
- In D-vines one can select more freely which pairs to model than in the canonical vine.

Inference

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- For general n and a specific vine decomposition the log likelihood $L(\Theta|x)$ can be computed using the density expressions for canonical and D-vines. Here x denotes the complete data and Θ the vector of all pairwise copula parameters.
- Algorithmic expressions are given in (Aas et al. 2007).

Inference for $n = 3$

$$
\begin{aligned}
L(\Theta|x) &= \sum_{t=1}^{T} [\log c_{13}(x_{1t}, x_{3t}|\Theta_{13}) + \log c_{23}(x_{2t}, x_{3t}|\Theta_{23}) \\
&+ \log c_{12|3}(v_{1t}, v_{2t}|\Theta_{1|23})]
\end{aligned}
$$

where

$$
\begin{aligned}
v_{1t} &:= F(x_{1t}|x_{3t}) = h(x_{1t}, x_{3t}, \Theta_{13}) \\
v_{2t} &:= F(x_{2t}|x_{3t}) = h(x_{2t}, x_{3t}, \Theta_{23}) \\
\Theta &:= (\Theta_{13}, \Theta_{23}, \Theta_{12|3})
\end{aligned}
$$

Starting values

- Estimate Θ_{13} from data $(x_{1,t}, x_{3,t}, t = 1, \cdots, T)$
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- Use these estimates to define

$$\begin{aligned} \hat{v}_{1t} &:= h(x_{1t}, x_{3t}, \hat{\Theta}_{13}) \\ \hat{v}_{2t} &:= h(x_{2t}, x_{3t}, \hat{\Theta}_{23}) \end{aligned}$$

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$$

- Estimate $\Theta_{12|3}$ from data $(\hat{v}_{1,t}, \hat{v}_{2,t}, t = 1, \cdots, T)$
- For Gauss and t-copula an estimate of Kendall's tau is used to estimate the correlation parameter, since this is independent of margins.
- Kendall's tau can also be used for the Clayton copula

Necessary Expressions

- For each pair-copula in the decomposition, three expressions are necessary:
  - bivariate density
  - h-function $h(x, v, \Theta) = F(x|v) = \frac{C_{x,v}(x,v,\Theta)}{\partial v}$
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  - inverse of h function wrt to x
- For Clayton this is available in closed form.
- For the Gumbel, Gaussian, Student's the inverse of the h-function must be obtained numerically.

Application: Financial Returns

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- *n*-dimensional Student's t-copula has been often used for modeling financial returns, however it only has a single parameter for tail dependence.
- Pair-copula constructions allows for multiple parameters for modeling tail dependence.

Data set

- Daily data from Jan. 4, 1999 until July 8, 2003 for

T	=	Norwegian stock index (TOTX)
M	=	MSCI world stock index
B	=	Norwegian bond index (BRIX)
S	=	SSBWG hedged bond index

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- Standardized residuals of a AR(1)-GARCH(1,1) model for the log return $x_{i,t}$

$$\begin{aligned} x_{i,t} &= c_i + \alpha_i\, x_{i,t-1} + \sigma_{i,t}\, z_{i,t}, \\ \mathrm{E}[z_{t,i}] &= 0 \text{ and } \mathrm{Var}[z_{t,i}] = 1, \\ \sigma^2_{i,t} &= a_{i,0} + a_i\, \epsilon^2_{i,t-1} + b_i\, \sigma^2_{i,t-1} \text{ where } \epsilon_{i,t-1} = \sigma_{i,t}\, z_{i,t} \end{aligned}$$

  are found to be independent and are transformed to uniform margins

Data set

- Fitted degree of freedom for a t-copula to each pair

Between	M	T	B
S	4.21	34.16	14.47
M		8.03	15.48
T			12.60

Which PCC?

Strongest dependence between S and M, M and T and T and B, which will be used in the top tree of a D-vine

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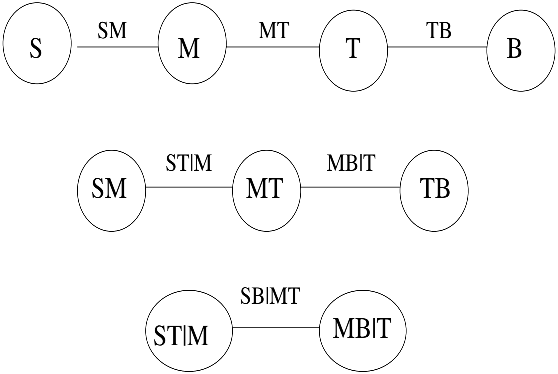

Parameter estimates

Param	*Start*	*Final*
ρ_{SM}	-0.25	-0.25
ρ_{MT}	0.47	0.47
ρ_{TB}	-0.17	-0.17
$\rho_{ST\|M}$	-0.11	-0.11
$\rho_{MB\|T}$	0.02	0.03
$\rho_{SB\|MT}$	0.29	0.28
ν_{SM}	4.21	4.34
ν_{MT}	16.65	16.26
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- cond. distribution of MSCI and BRIX given TOTX is close to normal

Comparison with multivariate Student's t-copula

- AIC

4 dim. Student's t-copula	-512.33
4 dim. Student's t pair copula decomposition	-487.42

[1] Can be used approximately since the 4 dim. Student's t-copula is nested within the 4 dim. Student's pair copula decomposition for large df's

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- Likelihood ratio test statistic [1]
  - 2 * Likelihood difference is 34.92 with 5 df
  - P-value is $1.5610^{-6} \Rightarrow$ 4 dim. Student's t-copula is rejected in favor of the PCC.

[1] Can be used approximately since the 4 dim. Student's t-copula is nested within the 4 dim. Student's pair copula decomposition for large df's

Tail dependence

- Upper and lower tail dependence coefficient for bivariate t-copula

$$\lambda_l(X,Y) = \lambda_u(X,Y) = 2\, t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$$

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$$\lambda_l(X,Y) = \lambda_u(X,Y) = 2\, t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$$

- Estimated tail dependence coefficients

Margin	*PCC*	*Student's t-copula*
SM	0.0279	0.0001
MT	0.0229	0.0317
TB	0.0005	0.0003

The probability of a joint large portfolio loss in SSBWG and BRIX stocks is much larger in the 4 dim. Student's t PCC compared to a 4 dim. Student's t-copula.

Bayesian model selection among PCC's

joint work with Dr. A. Min

- If a single pair copula type is used in a PCC of dimension n, then all other possible PCC's of dimension n provide a factorization of the same joint density

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- So we want to find reduced PCC's which best fits the data, i.e. let the data discover conditional independence conditions.
- $n = 3$:

Full PCC's:	$c_{12}c_{23}c_{13\|2} = c_{13}c_{23}c_{12\|3} = c_{12}c_{13}c_{23\|1}$
Reduced by 1 pair-copula:	$c_{12}c_{23},\ c_{12}c_{13\|2}$ $c_{13}c_{23},\ c_{23}c_{12\|3},\ c_{13}c_{12},\ c_{13}c_{23\|1}$
Reduced by 2 pair copulas:	$c_{12},\ c_{13},\ c_{23}$ $c_{13\|2},\ c_{12\|3},\ c_{23\|1}$

- Identifiability: Note that $c_{12}c_{23|1} = c_{23}c_{12|3}$ since $c_{12}c_{23|1} = c_{12}c_{13}c_{23|1} = c_{13}c_{23}c_{12|3} = c_{23}c_{12|3}$ when $c_{13} = 1$.

Bayesian framework of PCC's with $n \leq 3$

- Each of the 13 different PCC's with $n \leq 3$ is identified by a model index $\mathbf{m} = (m_1, m_2, m_3) = (i_1 j_1, i_2, j_2, i_3 j_3 | k_3)$.
  Example: $c_{12} c_{23|1}$ corresponds to $\mathbf{m} = (12, 00, 23|1)$.
- Each PCC model $\mathbf{m}$ has parameter vector $\boldsymbol{\theta}_m = (\boldsymbol{\theta}_{m_1}, \boldsymbol{\theta}_{m_2}, \boldsymbol{\theta}_{m_3})$.
  Example: Model $\mathbf{m} = (12, 23, 00|0)$ has parameter vector $\boldsymbol{\theta}_m = (\boldsymbol{\theta}_{12}, \boldsymbol{\theta}_{23}, \boldsymbol{\theta}_{00|0}) = (\boldsymbol{\theta}_{12}, \boldsymbol{\theta}_{23})$.

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- Goal is to estimate the best fitting model $\mathbf{m}$ and the corresponding parameter vector $\boldsymbol{\theta}_m$ using a Bayesian approach, i.e. the model $\mathbf{m}$ and θ_m are considered random quantities.
- Inference about $\mathbf{m}$ and θ_m is done via the joint posterior distribution of $(\mathbf{m}, \boldsymbol{\theta}_m)$.

MCMC algorithm for PCC's with $n \leq 3$

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  - Want to facilitate estimation without having to fit all models

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- Requirement: Need to accommodate varying model dimension, i.e. construct MCMC iterates $\boldsymbol{\theta}_{\mathbf{m}^r} = (\boldsymbol{\theta}^r_{m^r_1}, \boldsymbol{\theta}^r_{m^r_2}, \boldsymbol{\theta}^r_{m^r_3})$ where $\mathbf{m}^r$ is the current model at iteration r.

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- Solution: Reversible jump (RJ) MCMC proposed by (Green 1995)

General RJ MCMC (1)

- algorithm stays in current model using a Metropolis Hastings (MH) step
- algorithm moves to a larger model using a MH step (birth)
- algorithm moves to a smaller model using a MH step (death)

General RJ MCMC (2)

Model	M_1		M_2
Parameter	$\boldsymbol{\theta}^{(1)} \in \mathbf{R}^{d_1}$		$\boldsymbol{\theta}^{(2)} \in \mathbf{R}^{d_2}$
	d_1	$<$	d_2
Proposal	$\boldsymbol{\eta}^{(1)} \sim \varphi_1(\cdot)$		$\boldsymbol{\eta}^{(2)} \sim \varphi_2(\cdot)$
	$\boldsymbol{\eta}^{(1)} \in \mathbf{R}^{a_1}$		$\boldsymbol{\eta}^{(2)} \in \mathbf{R}^{a_2}$
Dimension matching	$d_1 + a_1$	$=$	$d_2 + a_2$
Bijection	$\begin{pmatrix} \boldsymbol{\theta}^{(1)} \\ \boldsymbol{\eta}^{(1)} \end{pmatrix}$	$\leftrightarrow$	$\begin{pmatrix} \boldsymbol{\theta}^{(2)} \\ \boldsymbol{\eta}^{(2)} \end{pmatrix}$

Acceptance probability for MH step from M_1 to M_2

$$\alpha(\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}) = \min\{1, A\}, \text{ where}$$

$$A := \frac{p(2, \boldsymbol{\theta}^{(2)}|\mathbf{y})}{p(1, \boldsymbol{\theta}^{(1)}|\mathbf{y})} \cdot \frac{p_{2\to1}}{p_{1\to2}} \cdot \frac{\varphi_2(\boldsymbol{\eta}^{(2)})}{\varphi_1(\boldsymbol{\eta}^{(1)})} \cdot \left|\frac{\partial\left(\boldsymbol{\theta}^{(2)}, \boldsymbol{\eta}^{(2)}\right)}{\partial\left(\boldsymbol{\theta}^{(1)}, \boldsymbol{\eta}^{(1)}\right)}\right|$$

$p(k, \boldsymbol{\theta}^{(k)}|y)$ = joint posterior density of M_k and $\boldsymbol{\theta}^{(k)}$

$p_{i\to j}$ = prior switching probability from M_i to M_j

$\varphi_k(\boldsymbol{\eta}^{(k)})$ = proposal distribution of $\boldsymbol{\eta}^{(k)}$ (suitably chosen)

$\left|\frac{\partial\left(\boldsymbol{\theta}^{(2)},\boldsymbol{\eta}^{(2)}\right)}{\partial\left(\boldsymbol{\theta}^{(1)},\boldsymbol{\eta}^{(1)}\right)}\right|$ = Jacobian of bijection

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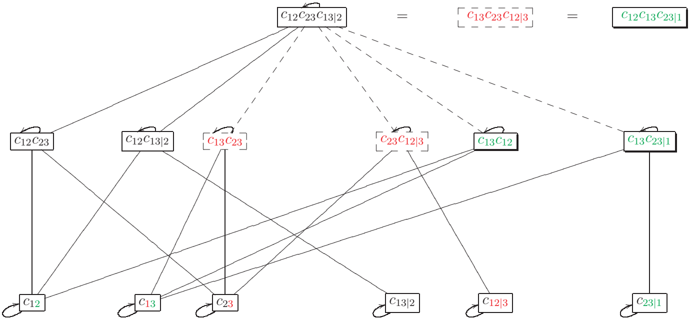
For moves from M_2 to M_1 we use A^{-1}.

RJ MCMC moves for selecting PCC's (n=3)

Notation	Meaning	Example
B0D0	stay move	$c_{12} \longrightarrow c_{12}$
B1D0	birth of 1 factor	$c_{12} \longrightarrow c_{12}\,c_{23}$
D1B0	death of 1 factor	$c_{12}\,c_{23} \longrightarrow c_{12}$
B2D1	birth of 2 factors death of 1 factor	$c_{13}\,c_{23} \longrightarrow c_{12}\,c_{23}\,c_{13\|2}$
D2B1	death of 2 factors birth of 1 factor	$c_{12}\,c_{23}\,c_{13\|2} \longrightarrow c_{13}\,c_{23}$
B3D2	birth of 3 factors death of 2 factors	$c_{13}\,c_{23\|1} \longrightarrow c_{12}\,c_{23}\,c_{13\|2}$
D3B2	death of 3 factors birth of 2 factors	$c_{12}\,c_{23}\,c_{13\|2} \longrightarrow c_{13}\,c_{23\|1}$

Graph of all PCC's with $n \leq 3$

Solid lines: B0D0, B1D0 and D1B0 moves
Dotted lines: B2D1, D2B1, B3D2 and D3B2 moves



Priors and proposal distributions

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- We assume equal prior switching probabilities for staying, moving to a smaller and moving to a larger model
- Independent normal distributions are assumed for each η component centered at corresponding previous parameter component with specified variance

Likelihood for PCC's with $n \leq 3$

Assume bivariate t-copula for each pair copula term, i.e.

$$
\begin{aligned}
\boldsymbol{\theta} &= (\rho, \nu) \quad \rho = \textbf{correlation} \ , \nu = \ \textbf{df} \\
c(u_1, u_2 | \boldsymbol{\theta}) &= \frac{\Gamma(\frac{\nu+2}{2})/\Gamma(\frac{\nu}{2})}{\nu\pi t_\nu(x_1) t_\nu(x_2)\sqrt{1-\rho^2}} \left(1 + \frac{x_1^2 + x_2^2 - 2\rho\, x_1\, x_2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+1}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= t_\nu^{-1}(u_1) \\
x_2 &= t_\nu^{-1}(u_2) \\
t_\nu(\cdot) &= \text{pdf of univariate } t_\nu \text{ distribution} \\
t_\nu^{-1}(\cdot) &= \text{quantile function of } t_\nu \text{ distribution}
\end{aligned}
$$

Acceptance probability for B2D1 move

Actual state	$\theta^o := \theta^o_{\mathbf{m}^o}$	$\mathbf{m}^o = (m^o_1, m^o_2, m^o_3)$
Proposed state	$\theta^p := \theta^p_{\mathbf{m}^p}$	$\mathbf{m}^p = (m^p_1, m^p_2, m^p_3)$

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Proposed state	$\theta^p := \theta^p_{\mathbf{m}^p}$	$\mathbf{m}^p = (m^p_1, m^p_2, m^p_3)$
birth indices	$\{m^p_s, m^p_t\} :=$	$\mathbf{m}^p \backslash \mathbf{m}^o$
death index	$m^o_v :=$	$\mathbf{m}^o \bigcap \mathbf{m}^p$
stay index	$m^o_w :=$	$\mathbf{m}^o \backslash \mathbf{m}^p$

Example: $(\theta^o_{13}, \theta^o_{23}) \rightarrow (\theta^p_{12}, \theta^p_{23}, \theta^p_{13|2})$

Acceptance probability for B2D1 move

Actual state	$\theta^o := \theta^o_{\mathbf{m}^o}$	$\mathbf{m}^o = (m^o_1, m^o_2, m^o_3)$
Proposed state	$\theta^p := \theta^p_{\mathbf{m}^p}$	$\mathbf{m}^p = (m^p_1, m^p_2, m^p_3)$
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Example: $(\theta^o_{13}, \theta^o_{23}) \rightarrow (\theta^p_{12}, \theta^p_{23}, \theta^p_{13|2})$ **Bijection:**

$$\begin{pmatrix} \theta^p \\ \eta^p_{m^o_w} \end{pmatrix} := \begin{pmatrix} \theta^p_{m^p_s} \\ \theta^p_{m^p_t} \\ \theta^p_{m^o_v} \\ \eta^p_{m^o_w} \end{pmatrix} = \begin{pmatrix} \eta^o_{m^p_s} \\ \eta^o_{m^p_t} \\ \theta^o_{m^o_v} \\ \theta^o_{m^o_w} \end{pmatrix} =: \begin{pmatrix} \eta^o_{m^p_s} \\ \eta^o_{m^p_t} \\ \theta^o \end{pmatrix}$$

Acceptance probability for B2D1 move

Actual state	$\theta^o := \theta^o_{\mathbf{m}^o}$	$\mathbf{m}^o = (m_1^o, m_2^o, m_3^o)$
Proposed state	$\theta^p := \theta^p_{\mathbf{m}^p}$	$\mathbf{m}^p = (m_1^p, m_2^p, m_3^p)$
birth indices	$\{m_s^p, m_t^p\} :=$	$\mathbf{m}^p \backslash \mathbf{m}^o$
death index	$m_v^o :=$	$\mathbf{m}^o \bigcap \mathbf{m}^p$
stay index	$m_w^o :=$	$\mathbf{m}^o \backslash \mathbf{m}^p$

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$$\begin{pmatrix} \theta^p \\ \eta^p_{m_w^o} \end{pmatrix} := \begin{pmatrix} \theta^p_{m_s^p} \\ \theta^p_{m_t^p} \\ \theta^p_{m_v^o} \\ \eta^p_{m_w^o} \end{pmatrix} = \begin{pmatrix} \eta^o_{m_s^p} \\ \eta^o_{m_t^p} \\ \theta^o_{m_v^o} \\ \theta^o_{m_w^o} \end{pmatrix} =: \begin{pmatrix} \eta^o_{m_s^p} \\ \eta^o_{m_t^p} \\ \theta^o \end{pmatrix}$$

Here $\eta^o_{m_s^p}$ and $\eta^o_{m_t^p}$ are bivariate normal centered at $\theta_{m_s^p, last}$, and $\theta_{m_t^p, last}$ with specified covariance matrix, while we set $\eta^p_{m_w^o} = \theta^o_{m_w^o}$

First results for selecting between $c_{12}c_{23}c_{13|2}$ and $c_{12}c_{23}$

5000 data points generated from $c_{12}c_{23}c_{13|2}$ with every pair copula term a bivariate t-copula
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	Full model				Reduced model			
Parameter	**mean**	**median**	**5%**	**95%**	**mean**	**median**	**5%**	**95%**
$\rho_{12} = .2$	0.19	0.19	0.17	0.21	0.19	0.19	0.16	0.22
$\nu_{12} = 5$	5.07	4.97	4.39	6.17	5.23	5.19	4.47	6.04
$\rho_{23} = 0.6$	0.60	0.60	0.58	0.61	0.60	0.60	0.59	0.62
$\nu_{23} = 7$	7.58	7.44	6.38	9.40	8.01	7.85	6.51	10.06
$\rho_{13\|2} = 0.2$	0.19	0.19	0.13	0.23				
$\nu_{13\|2} = 40$	13.35	13.20	11.88	15.62				

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