Optimal Dynamic Hedging Using Copula-Threshold-GARCH Models

YiHao Lai
College of Business
Feng Chia University, Taiwan

Cathy Chen
Statistics and Actuarial Science
Feng Chia University, Taiwan

Richard Gerlach
Econometrics and Business Statistics
University of Sydney, Australia

Outline

- Hedging Strategies and the Optimal Hedge Ratio
- Bivariate models
- Marginals
- Copulas
- Data and Empirical Results
- Hedge performance comparison
- Conclusions

Hedge Strategies and Optimal Hedge Ratio

$$R_t^P = R_t^S - \beta R_t^F$$

1. **One-to-one** strategy

- Set $\beta = 1$
- Works when both markets are highly (perfectly) correlated

2. Variance minimisation

$$Var(R_t^P) = Var(R_t^S) + \beta^2 Var(R_t^F) - 2\beta Cov(R_t^S, R_t^F)$$

Optimal hedge ratio

$$\beta = \frac{Cov(R_t^S, R_t^F)}{Var(R_t^F)}$$

Constant optimal hedge ratio: OLS

$$R_t^S \approx \alpha + \beta^{OLS} R_t^F$$

Correlation Analysis

$$\beta = \rho \sqrt{\frac{Var(R_t^S)}{Var(R_t^F)}}$$

Time-varying optimal hedge ratio:

$$\beta_t = \rho_t \frac{\sigma_t^S}{\sigma_t^F}$$

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Time varying correlations

Bivariate volatility (GARCH) models

1. Dynamic conditional correlation – *DCC (Engle, 2002)*

$$\tilde{R}_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} = D_t \eta_t \quad ; \quad D_t = \begin{bmatrix} \sqrt{h_{1,t}} & 0 \\ 0 & \sqrt{h_{2,t}} \end{bmatrix}$$

$$\eta_t \sim N_2(0, \mathrm{P}_t)$$

$$h_{1,t} = \alpha_0 + \alpha_1 \varepsilon_{1,t-1}^2 + \alpha_2 h_{1,t-1}$$

$$h_{2,t} = \beta_0 + \beta_1 \varepsilon_{2,t-1}^2 + \beta_2 h_{2,t-1}$$

$$\mathrm{P}_t = Q_t^{*-1} Q_t Q_t^{*-1} \qquad H_t = D_t \mathrm{P}_t D_t$$

$$Q_t = \bar{\rho}(1 - \varpi_1 - \varpi_2) + \varpi_1 \eta_{t-1} \eta_{t-1}^T + \varpi_2 Q_{t-1}$$

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Copula dependence

Bivariate copula dependence structure
Sklar (1959)

$$g(R^S, R^F) = c\left(F^S(R^S), F^F(R^F)\right) \times f_S(R^S) \times f_F(R^F)$$

Patton (2001) - copula time series

$$g(R_t^S, R_t^F \mid I_{t-1}) = c_t\left(F_t^S(R_t^S \mid I_{t-1}), F_t^F(R_t^F \mid I_{t-1}) \middle| I_{t-1}\right) \times f_t^S(R_t^S \mid I_{t-1}) \times f_t^F(R_t^F \mid I_{t-1}),$$

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Time varying correlations

Bivariate volatility (GARCH) models

2. Threshold GARCH-t marginal structure

$$R_t^S = \phi_0 + \phi_1 R_{t-1}^S + \phi_2 R_{t-1}^F + \varepsilon_t^S$$

$$h_t^S = \begin{cases} \alpha_0^{(1)} + \alpha_1^{(1)} \varepsilon_{t-1}^2 + \alpha_2^{(1)} h_{t-1}^S, & \varepsilon_{t-1}^S < 0 \\ \alpha_0^{(2)} + \alpha_1^{(2)} \varepsilon_{t-1}^2 + \alpha_2^{(2)} h_{t-1}^S, & \varepsilon_{t-1}^S \geq 0 \end{cases}$$

$$\varepsilon_t^S \mid I_{t-1} \sim t_{v^S}(0, h_t^S)$$

Black (1976) → So, Chen and Chen (2005)

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Time varying correlations

Futures returns marginal

$$R_t^F = \varphi_0 + \varphi_1 R_{t-1}^S + \varphi_2 R_{t-1}^F + \eta_t$$

$$h_t^F = \begin{cases} \beta_0^{(1)} + \beta_1^{(1)} \varepsilon_{t-1}^2 + \beta_2^{(1)} h_{t-1}^S, & \eta_{t-1} < 0 \\ \beta_0^{(2)} + \beta_1^{(2)} \varepsilon_{t-1}^2 + \beta_2^{(2)} h_{t-1}^S, & \eta_{t-1} \geq 0 \end{cases}$$

$$\eta_t \mid I_{t-1} \sim t_{\nu^F}(0, h_t^F;)$$

We allow an explosive volatility process:

Medeiros and Veiga (2005), Wong and Li (2001), Gerlach and Chen (2007)

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Copula-Threshold-GARCH Model [Cont.]

Copulas considered

- Gaussian
- Student-t
- Clayton (1978)
- Gumbel (1960)
- Mixture of Clayton and Gumbel (Li, 2000; Hu, 2003)

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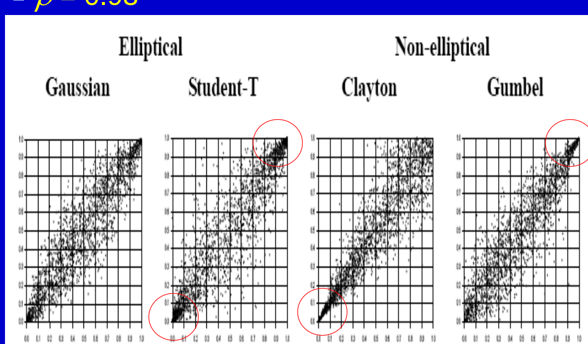
Copulas and Tail dependence

- ρ = 0.93



$$c_t^{Mixture}(u_t, v_t; q, \delta^C, \delta^G) = q c_t^{Clayton}(u_t, v_t; \delta^C) + (1-q) c_t^{Gumbel}(u_t, v_t; \delta^G)$$

Tail dependence

Copula	$\lim_{u \to 0} \Pr\left(X \leq F_X^{-1}(u) \mid Y \leq F_Y^{-1}(u)\right)$	$\lim_{u \to 1} \Pr\left(X \geq F_X^{-1}(u) \mid Y \geq F_Y^{-1}(u)\right)$
Gaussian	0	0
Student's t	$2T_{\nu_c+1}\left(-\sqrt{\nu_c+1}\frac{\sqrt{1-\rho_t}}{\sqrt{1+\rho_t}}\right)$	$2T_{\nu_c+1}\left(-\sqrt{\nu_c+1}\frac{\sqrt{1-\rho_t}}{\sqrt{1+\rho_t}}\right)$
Clayton	$2^{-1/\delta^C}$	0
Gumbel	0	$2 - 2^{1/\delta^G}$

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Elliptical Copulas

- **Gaussian**

$$C_t^{Gaussian}(u_t, v_t; \rho_t) = \Phi_2\left(\Phi^{-1}(u_t), \Phi^{-1}(v_t); \rho_t\right) = \Phi_2\left(a_t, b_t; \rho_t\right)$$

$$c_t^{Gaussian}(u_t, v_t; \rho_t) = \frac{\phi_2\left(a_t, b_t; \rho_t\right)}{\phi\left(a_t\right)\phi\left(b_t\right)}$$

- **Student's t**

$$C_t^{Student}(u_t, v_t; \rho_t) = T_{2;\nu_c}\left(T_{\nu_c}^{-1}(u_t), T_{\nu_c}^{-1}(v_t)\right) = T_{2;\nu_c}\left(a_t, b_t\right)$$

$$c_t^{Student}(u_t, v_t; \rho_t) = \frac{t_{2;\nu_c}\left(a_t, b_t; \rho_t\right)}{t_{\nu_c}\left(a_t\right)t_{\nu_c}\left(b_t\right)}$$

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Archimedean Copulas

- Clayton

$$C_t^{Clayton}(u_t, v_t; \delta^C) = (u_t^{-\delta} + v_t^{-\delta} - 1)^{-1/\delta}$$

$$c_t^{Clayton}(u_t, v_t; \delta^C) = (1+\delta)(u_t v_t)^{-1-\delta}(u_t^{-\delta} + v_t^{-\delta} - 1)^{-2-1/\delta}$$

- Gumbel

$$C_t^{Gumbel}(u_t, v_t; \delta^G) = \exp\left\{-\left[\left(-\ln(u_t)\right)^{\delta^G} + \left(-\ln(v_t)\right)^{\delta^G}\right]^{\frac{1}{\delta^G}}\right\}$$

$$c_t^{Gumbel}(u_t, v_t; \delta^G) = \frac{\exp\left\{-\left[\left(-\ln(u_t)\right)^{\delta^G} + \left(-\ln(v_t)\right)^{\delta^G}\right]^{\frac{1}{\delta^G}}\right\}\left(\ln(u_t)\ln(v_t)\right)^{\delta^G-1}\left\{\left[\left(-\ln(u_t)\right)^{\delta^G} + \left(-\ln(v_t)\right)^{\delta^G}\right]^{\frac{1}{\delta^G}} + \delta^G - 1\right\}}{u_t v_t\left[\left(-\ln(u_t)\right)^{\delta^G} + \left(-\ln(v_t)\right)^{\delta^G}\right]^{2-\frac{1}{\delta^G}}}$$

Time-varying linear dependence

Patton (2001) → Bartram, Taylor and Wang (2004)

$$\rho_t = \kappa + \lambda_1 \rho_{t-1} + \gamma \zeta_t$$

$$\zeta_t = \lambda_2 \zeta_{t-1} + (1-\lambda_2)\,|u_{t-1} - v_{t-1}|$$

↓

$$\rho_t = \varphi + (\lambda_1 + \lambda_2)\rho_{t-1} - \lambda_1 \lambda_2 \rho_{t-2} + \pi\,|u_{t-1} - v_{t-1}|$$

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Clayton and Gumbel copulas

- Copula δ, Kendall's τ and ρ

$$\tau_t = (2/\pi)\sin^{-1}(\rho_t)$$

$$\delta_t^C = 2\tau_t/(1-\tau_t) \qquad \delta_t^G = 1/(1-\tau_t)$$

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Estimation

- Inference for margins, *IFM* (Joe and Xu, 1996)

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$$\hat{\theta}_S = \arg\max \sum_{t=1}^{T} \log f_t^S(r_t^S \mid I_{t-1}, \theta_S) \quad (2.5)$$
$$\hat{\theta}_F = \arg\max \sum_{t=1}^{T} \log f_t^F(r_t^F \mid I_{t-1}, \theta_F) \quad (2.6)$$

2
$$\hat{\theta}_c = \arg\max \sum_{t=1}^{T} \log c_t(u_t, v_t \mid I_{t-1}, \theta_c) \quad (2.7)$$

$$u_t = T_{v^S}\left(\frac{\varepsilon_t}{\sqrt{h_t^S}}\right) \qquad v_t = T_{v^F}\left(\frac{\eta_t}{\sqrt{h_t^F}}\right)$$

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Markets considered

- Spot price indices are
  1. Hong Kong: Hang Seng Price Index (*HNGKNGI*)
  2. Japan: Nikkei 225 Stock Average Price Index (*JAPDOWA*)
  3. Korea Stock Exchange Composite Price Index (*KORCOMP*)
  4. Singapore: Straits Times Price Index (*SNGPORI*)
  5. Taiwan Stock Exchange Capitalization weighted Stock Index (*TAIEX*)
- The corresponding futures stock price indices are
  1. Hong Kong Futures Exchange Hang Seng Index (*HSI*)
  2. Japan: Osaka Stock Exchange Nikkei 225 Index (*ONA*)
  3. Korea Stock Exchange KOSPI 200 Index (*KKX*)
  4. Singapore International Monetary Exchange Straits Times Index (*SST*)
  5. Taiwan Futures Exchange Index (*TAIFEX*).
- Sample period: January 1, 1998 - June 10, 2005

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Estimates from TGARCH – spot returns

	Hong Kong		Japan		Korea		Singapore		Taiwan	
Variable	Coeff	Std	Coeff	Std	Coeff	Std	Coeff	Std	Coeff	Std
ϕ_0	0.01	(0.03)	-0.01	(0.03)	0.07	(0.04)	0.016	(0.024)	-0.003	(0.03)
ϕ_1	**-0.22**	(0.06)			**-0.12**	(0.02)			-0.02	(0.07)
ϕ_2	0.24	(0.06)			0.15	(0.02)			0.09	(0.06)
$\alpha_0^{(1)}$					0.24	(0.01)				
$\alpha_1^{(1)}$	**0.06**	(0.01)	**0.08**	(0.02)	0.05	(0.003)	**0.07**	(0.022)	**0.11**	(0.03)
$\alpha_2^{(1)}$	**0.995**	(0.02)	**0.97**	(0.02)	0.91	(0.003)	**0.997**	(0.021)	**0.98**	(0.02)
$\alpha_0^{(2)}$	0.02	(0.01)	0.05	(0.02)	**-0.23**	(0.01)	0.02	(0.011)	0.09	(0.04)
$\alpha_1^{(2)}$	0.02	(0.01)	0.03	(0.01)	0.03	(0.003)	0.03	(0.017)	0.035	(0.02)
$\alpha_2^{(2)}$	0.92	(0.02)	0.91	(0.02)	**1.02**	(0.002)	0.89	(0.033)	0.85	(0.04)
v^S	**5.80**	(0.80)	**7.39**	(1.15)	**5.82**	(0.58)	**6.89**	(1.076)	**10.66**	(2.6)
$Q(10)$	3.6 [0.96]		3.5 [0.97]		3.2 [0.98]		11.0 [0.35]		5.6 [0.85]	
$Q^2(10)$	12.7 [0.24]		5.0 [0.89]		13.6 [0.19]		4.9 [0.90]		16.1 [0.10]	

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Estimates from TGARCH – futures returns

	Hong Kong		Japan		Korea		Singapore		Taiwan	
Variable	Coeff	Std	Coeff	Std	Coeff	Std	Coeff	Std	Coeff	Std
ω_0	0.01	(0.03)	-0.01	(0.03)	0.09	(0.004)	0.03	(0.02)	0.01	(0.04)
ω_1			-0.15	(0.07)						
ω_2			0.10	(0.07)						
$\beta_0^{(1)}$					-0.04	(0.001)	-0.03	(0.02)		
$\beta_1^{(1)}$	0.06	(0.01)	0.08	(0.01)	0.05	(0.004)	0.09	(0.02)	0.11	(0.02)
$\beta_2^{(1)}$	1.0	(0.01)	0.98	(0.02)	1.01	(0.003)	1.01	(0.03)	0.97	(0.02)
$\beta_0^{(2)}$	0.02	(0.01)	0.06	(0.02)	0.08	(0.004)	0.05	(0.03)	0.10	(0.03)
$\beta_1^{(2)}$	0.01	(0.01)	0.03	(0.01)	0.05	(0.004)	0.05	(0.02)	0.05	(0.02)
$\beta_2^{(2)}$	0.92	(0.02)	0.89	(0.01)	0.89	(0.003)	0.85	(0.03)	0.86	(0.03)
v^F	5.69	(0.9)	6.96	(1.0)	6.26	(0.7)	4.34	(0.6)	5.26	(0.8)
$Q(10)$	5.4 [0.87]		2.4 [0.99]		2.6 [0.99]		13.2 [0.21]		10.1 [0.44]	
$Q^2(10)$	16.0 [0.10]		15.4 [0.11]		13.0 [0.22]		3.1 [0.98]		15.5 [0.12]	

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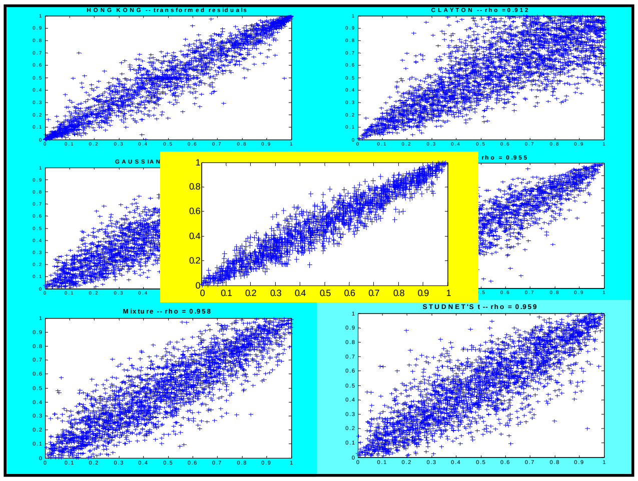

Tail dependence - HK

Copula	$\lim_{u\to 0}\Pr\left(X\le F_X^{-1}(u)\mid Y\le F_Y^{-1}(u)\right)$	$\lim_{u\to 1}\Pr\left(X\ge F_X^{-1}(u)\mid Y\ge F_Y^{-1}(u)\right)$
Gaussian	0	0
Student's t	0.70	0.70
Clayton	0.88	0
Gumbel	0	0.86

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$$\rho_t = \varphi + (\lambda_1 + \lambda_2)\rho_{t-1} - \lambda_1\lambda_2\rho_{t-2} + \pi\,|u_{t-1} - v_{t-1}| \quad (2.19)$$

		Gau	Stud-t	Clay	Gum	Mix
HK	φ	0.60	0.54 (*ν=4.8*)	0.45	0.1	0.47 (q=0.19)
	λ	0.38	0.45	0.52	0.90	0.51
	π	**-0.11**	**-0.13**	**-0.28**	**-0.08**	**-0.12**
Jap	φ	0.15	0.46 (*ν=6.0*)	0.45	0.40	0.49 (q=0.22)
	λ	0.88	0.95	0.95	0.89	0.95
	π	**-0.07**	**-0.13**	**-0.27**	**-0.15**	**-0.13**
Sin	φ	0.14	0.20 (*ν=4.0*)	0.17	0.16	0.21 (q=0.26)
	λ	0.86	0.80	0.83	0.84	0.79
	π	**-0.20**	**-0.13**	**-0.28**	**-0.22**	**-0.24**

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Descriptive statistics for ρ

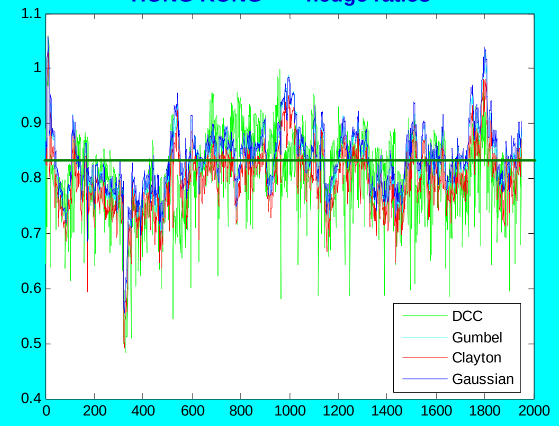
		Hong Kong	Japan	Korea	Singapore	Taiwan
Unconditional		0.946	0.952	0.906	0.909	0.930
Gaussian copula					volatile	
Mean		0.957	0.956	0.934	0.872	0.944
Maximum		0.968	0.973	0.978	0.976	0.971
Minimum		0.884	0.908	0.825	0.670	0.850
SD	Stable	0.008	0.008	0.027	0.052	0.015
Student-t copula						
Mean	max	0.959	0.957	0.938	0.874	0.945
Maximum		0.972	0.973	0.980	0.962	0.969
Minimum		0.877	0.884	0.865	0.668	0.869
SD	Stable	0.009	0.010	0.026	0.049	0.013
Clayton copula						
Mean	min	0.912	0.912	0.877	0.794	0.886
Maximum		0.951	0.943	0.948	0.917	0.917
Minimum		0.733	0.766	0.753	0.526	0.669
SD	volatile	0.022	0.020	0.043	0.065	0.028
Gumbel copula						
Mean		0.955	0.951	0.930	0.864	0.936
Maximum		0.979	0.970	0.978	0.966	0.968
Minimum		0.914	0.869	0.827	0.656	0.843
SD		0.011	0.011	0.029	0.053	0.017
Mixture copula						
Mean		0.958	0.956	0.938	0.871	0.941
Maximum		0.971	0.970	0.980	0.959	0.970
Minimum		0.880	0.884	0.866	0.643	0.855
SD	Stable	0.009	0.010	0.025	0.050	0.015
DCC						
Mean		0.954	0.952	0.932	0.889	0.932
Maximum		0.981	0.985	0.973	0.975	0.993
Minimum		0.796	0.849	0.784	0.440	0.737
SD	volatile	0.021	0.007	0.030	0.058	0.018

Summary of estimated hedge ratios

stable · max · Highly volatile · min · volatile · min

	Hong Kong	Japan	Korea	Singapore	Taiwan
Unconditional	0.835	0.926	0.764	0.840	0.816
Gaussian copula					
Mean	0.841	0.936	0.823	0.787	0.835
Maximum	1.056	1.135	1.244	1.273	1.080
Minimum	0.556	0.770	0.456	0.499	0.571
SD	0.059	0.061	0.124	0.089	0.086
Student-t copula					
Mean	0.843	0.937	0.827	0.789	0.836
Maximum	1.061	1.134	1.239	1.305	1.082
Minimum	0.554	0.766	0.478	0.477	0.576
SD	0.060	0.061	0.126	0.090	0.086
Clayton copula					
Mean	0.801	0.892	0.774	0.716	0.784
Maximum	1.027	1.080	1.160	1.157	1.034
Minimum	0.493	0.706	0.445	0.422	0.481
SD	0.060	0.060	0.126	0.088	0.084
Gumbel copula					
Mean	0.839	0.931	0.820	0.780	0.829
Maximum	1.059	1.125	1.241	1.272	1.071
Minimum	0.561	0.759	0.455	0.477	0.561
SD	0.060	0.060	0.125	0.089	0.086
Mixture copula					
Mean	0.842	0.935	0.827	0.786	0.833
Maximum	1.060	1.133	1.240	1.302	1.079
Minimum	0.554	0.765	0.478	0.459	0.570
SD	0.060	0.060	0.125	0.090	0.086
DCC					
Mean	0.838	0.932	0.822	0.802	0.825
Maximum	1.050	1.134	1.226	1.266	1.075
Minimum	0.482	0.766	0.425	0.314	0.559
SD	0.065	0.060	0.125	0.094	0.084

HONG KONG -- hedge ratios



Difference in mean return

$R_p^{copula} - R_p^{competing}$

			Mean	Difference in mean returns (the higher the better) Unhedged	1-to-1	OLS	DCC
Hong Kong							
	Best Copula	Gumbel	0.0071	-0.009	0.0076	0.0048	-0.0008
	Best Traditional	Unhedged	0.0164	X			
Japan							
	Best Copula	Normal	0.0013	0.0156	0.0013	0.0024	0.0002
	Best Traditional	DCC	0.0011				O
Korea							
	Best Copula	Clayton	0.0152	-0.0308	0.0233	0.0107	0.0021
	Best Traditional	Unhedged	0.0460	X			
Singapore							
	Best Copula	Mixture	0.0026	-0.0028	0.0024	0.0015	0.0004
	Best Traditional	Unhedged	0.0054	X			
Taiwan							
	Best Copula	Gumbel	0.0010	0.0149	0.0014	0.0038	0.00004
	Best Traditional	DCC	0.00096				O

Hedge -- difference in return variance

$(V_p^{copula} - V_p^{traditional}) / V_p^{traditional}$

			Variance	Percentage of variance reduction (the lower the better) Unhedged	1-to-1	OLS	DCC
Hong Kong							
	Best Copula	Normal	0.2806	-89.7%	-26.5%	-2.24%	-1.90%
	Best Traditional	DCC	0.2860				O
Japan							
	Best Copula	Normal	0.1942	-90.7%	-5.8%	0.01%	-0.20%
	Best Traditional	OLS	0.1942			X	
Korea							
	Best Copula	Mixture	0.8520	-82.8%	-32.0%	-2.34%	-0.68%
	Best Traditional	DCC	0.8578				O
Singapore							
	Best Copula	Student-t	0.2176	-82.2%	-12.6%	2.44%	-0.35%
	Best Traditional	OLS	0.2124			X	
Taiwan							
	Best Copula	Normal	0.38021	-203.3%	-26.0%	-1.99%	-0.25%
	Best Traditional	DCC	0.38115				O

Summary

- Gaussian and Mixture copulas provide the best risk reduction for Hong Kong, Taiwan and Korea
- OLS provides the best-performed hedge ratio for risk reduction for Japan and Singapore
- Copula strategies provide close to the best returns among all hedge strategies.
- All copulas (bar Clayton) out-perform DCC model in risk reduction in all markets

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