Pricing, hedging and calibrating credit from the top down

Kay Giesecke

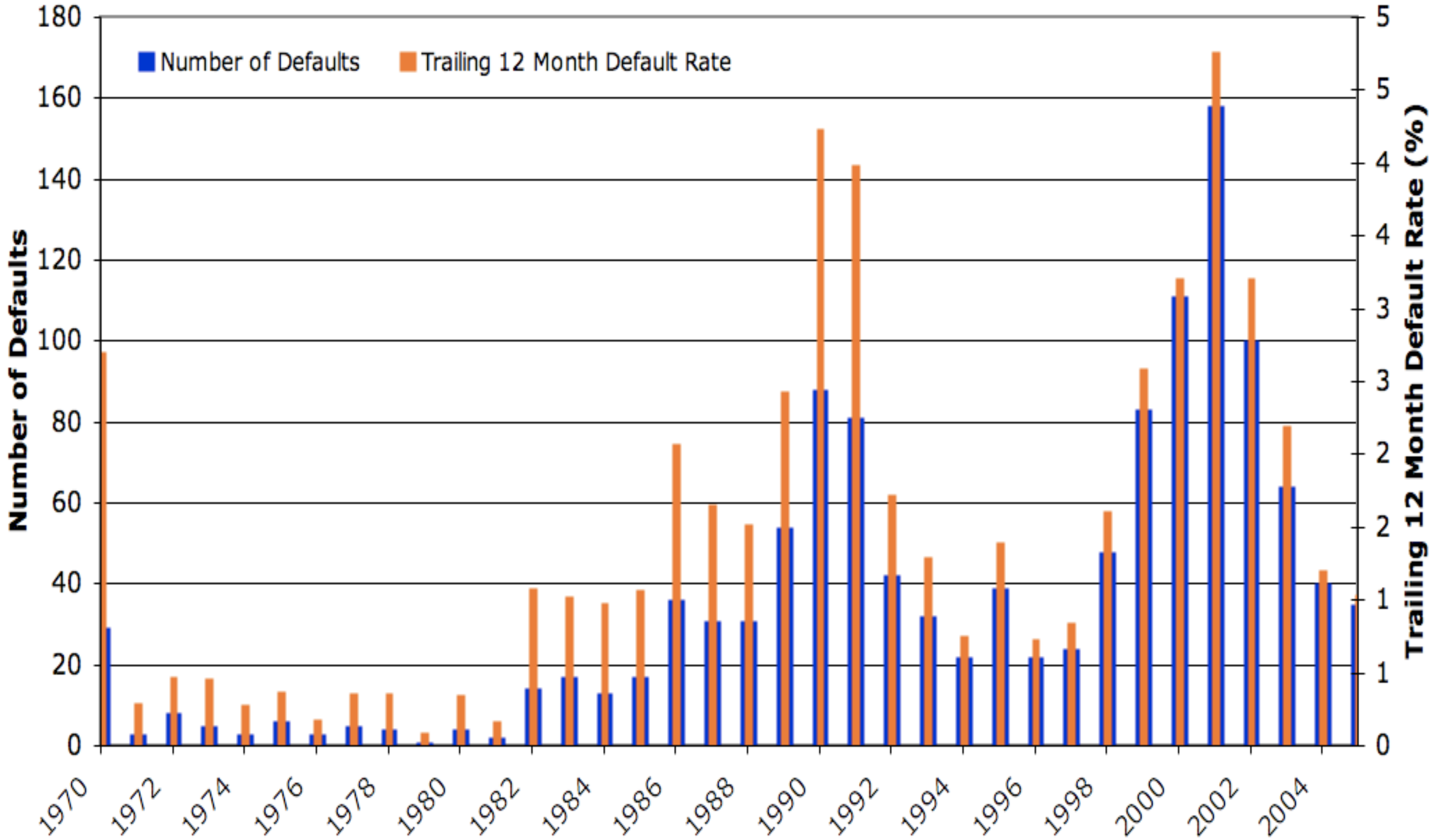
Management Science & Engineering

Stanford University

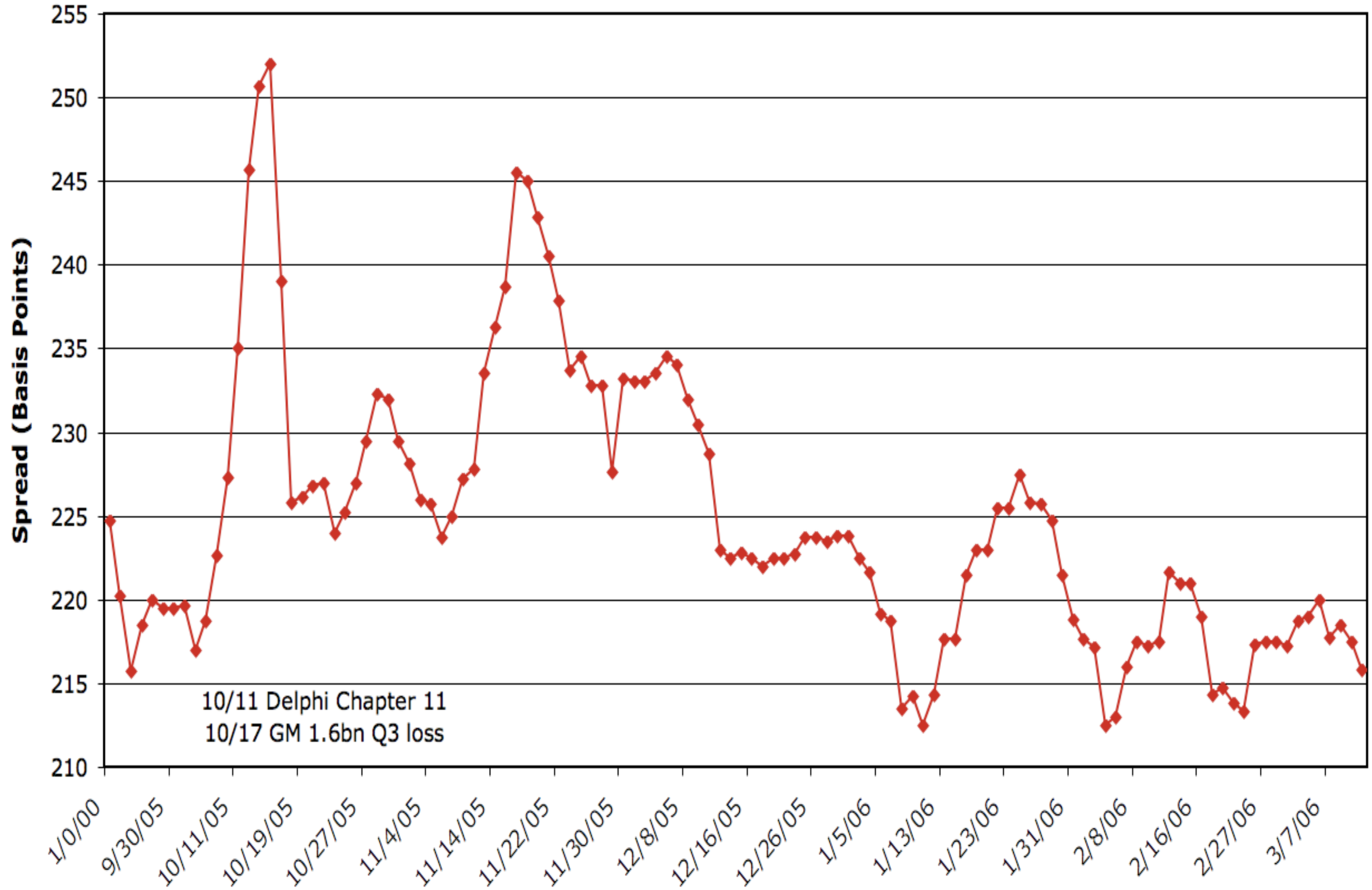
giesecke@stanford.edu

www.stanford.edu/∼giesecke

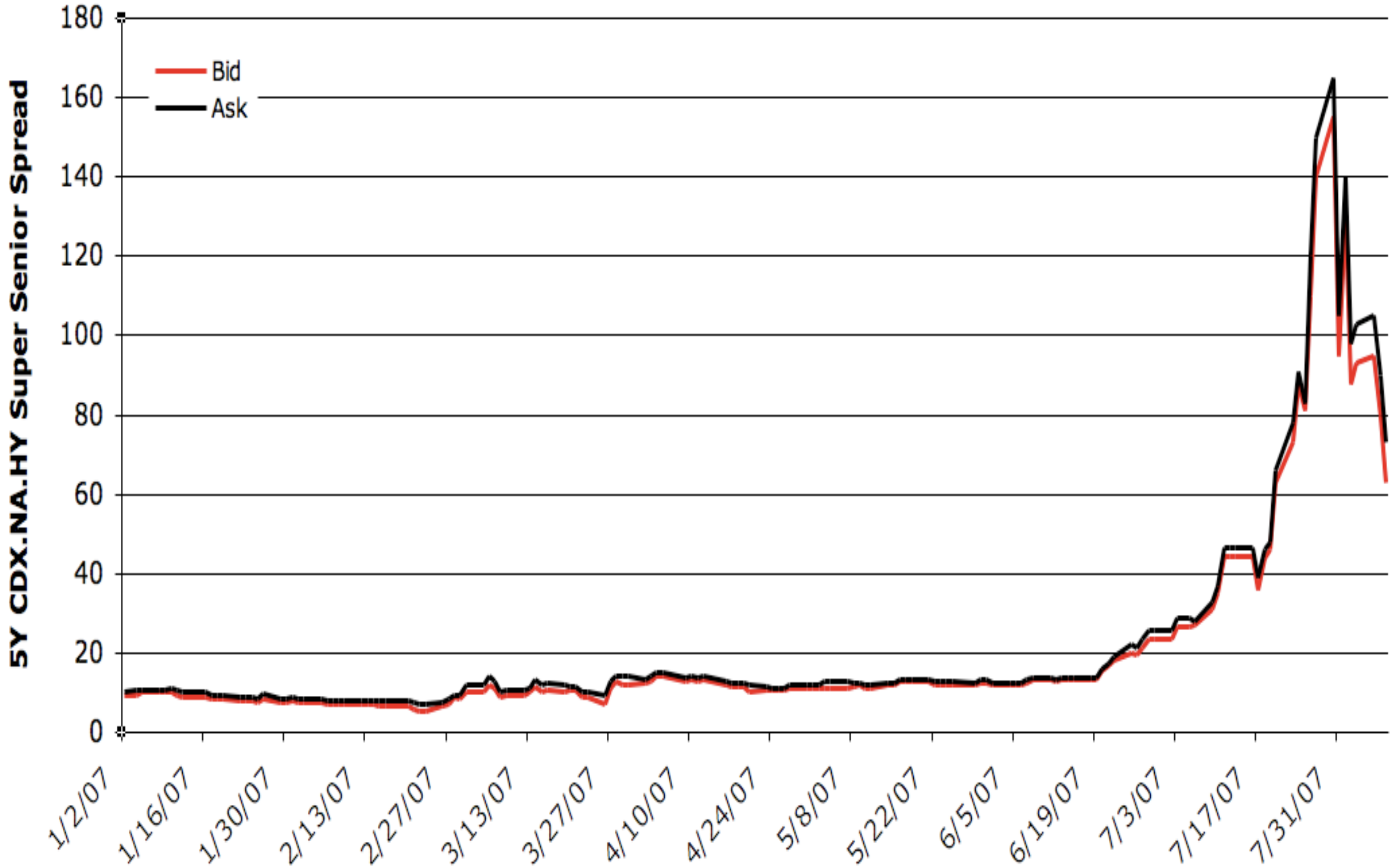
Defaults cluster: Moody's rated US issuers



Feedback in 5Y CDX.NA.XO Index



Feedback in 5Y CDX.NA.HY Super senior



Default dependence

- **Cyclical correlation**: first order effect
  - Firm sensitivity to common economic factors
- **Feedback or contagion**: second order effect
  - Complex web of business, informational and legal relationships
  - Important for contracts on loss volatility (e.g. tranche option)
  - Important for hedging of single name spread and default risk
  - Important for super senior risk

Portfolio credit derivative

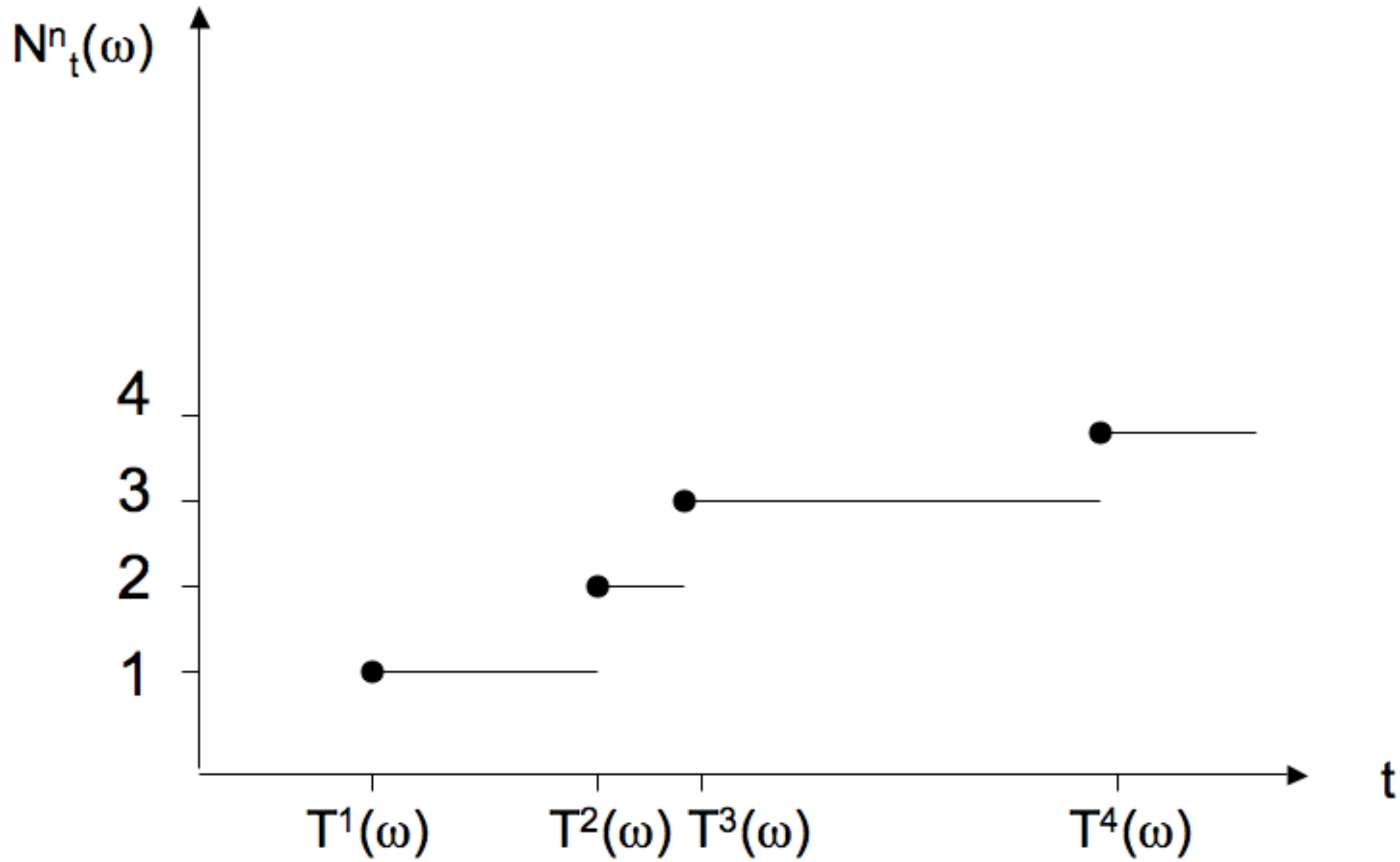
- Trading of default dependence in a portfolio of names
  - Targets portfolio loss or volatility of loss
- Contingent claim on portfolio loss
  - Defaults occur at stopping times $T^1 < T^2 < \ldots$
  - **Default process** N counts defaults

$$N_t = \sum_{k:T^k \leq t} 1$$

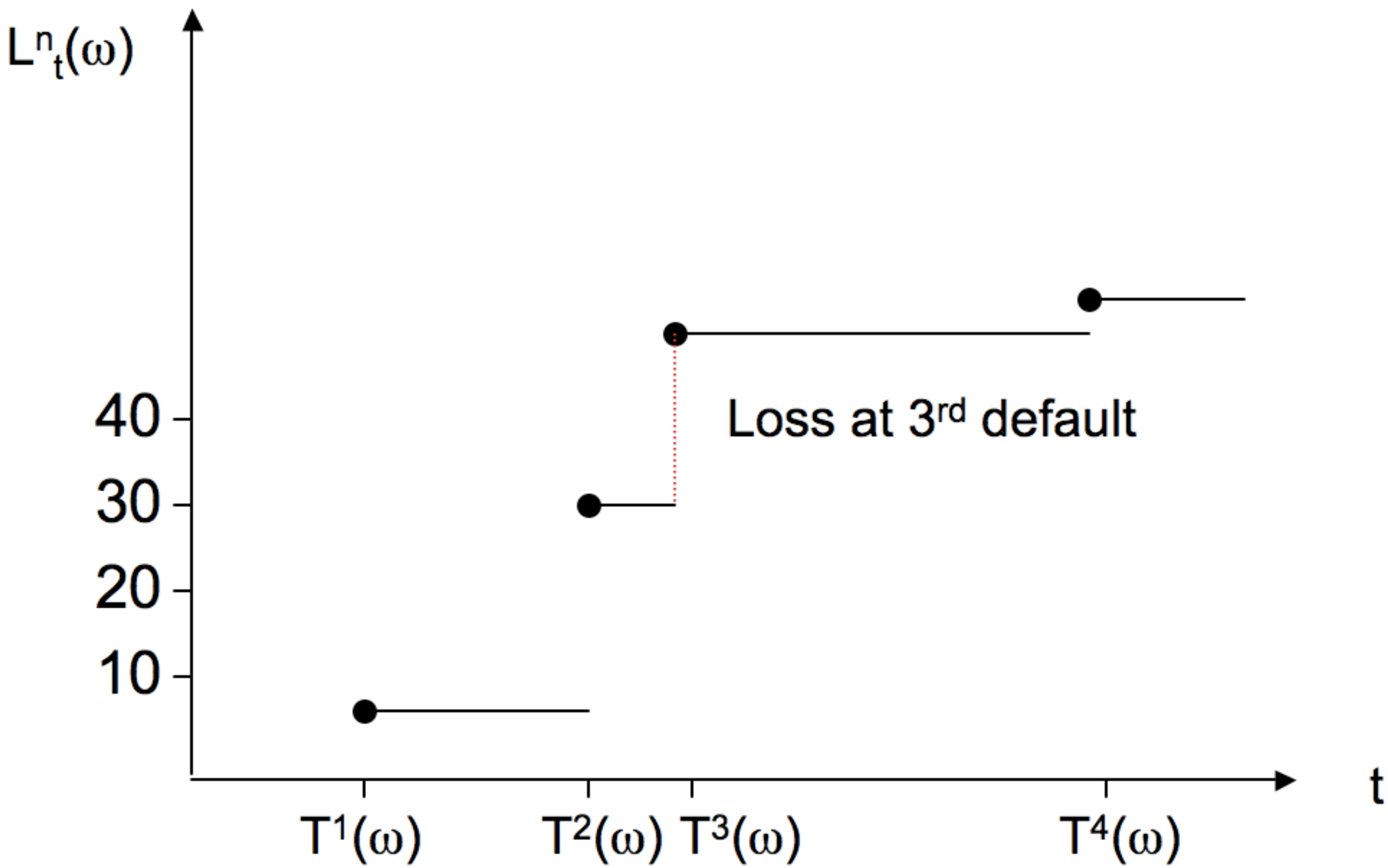
  - **Loss process** L records financial loss due to default

$$L_t = \sum_{k:T^k \leq t} \ell^k$$

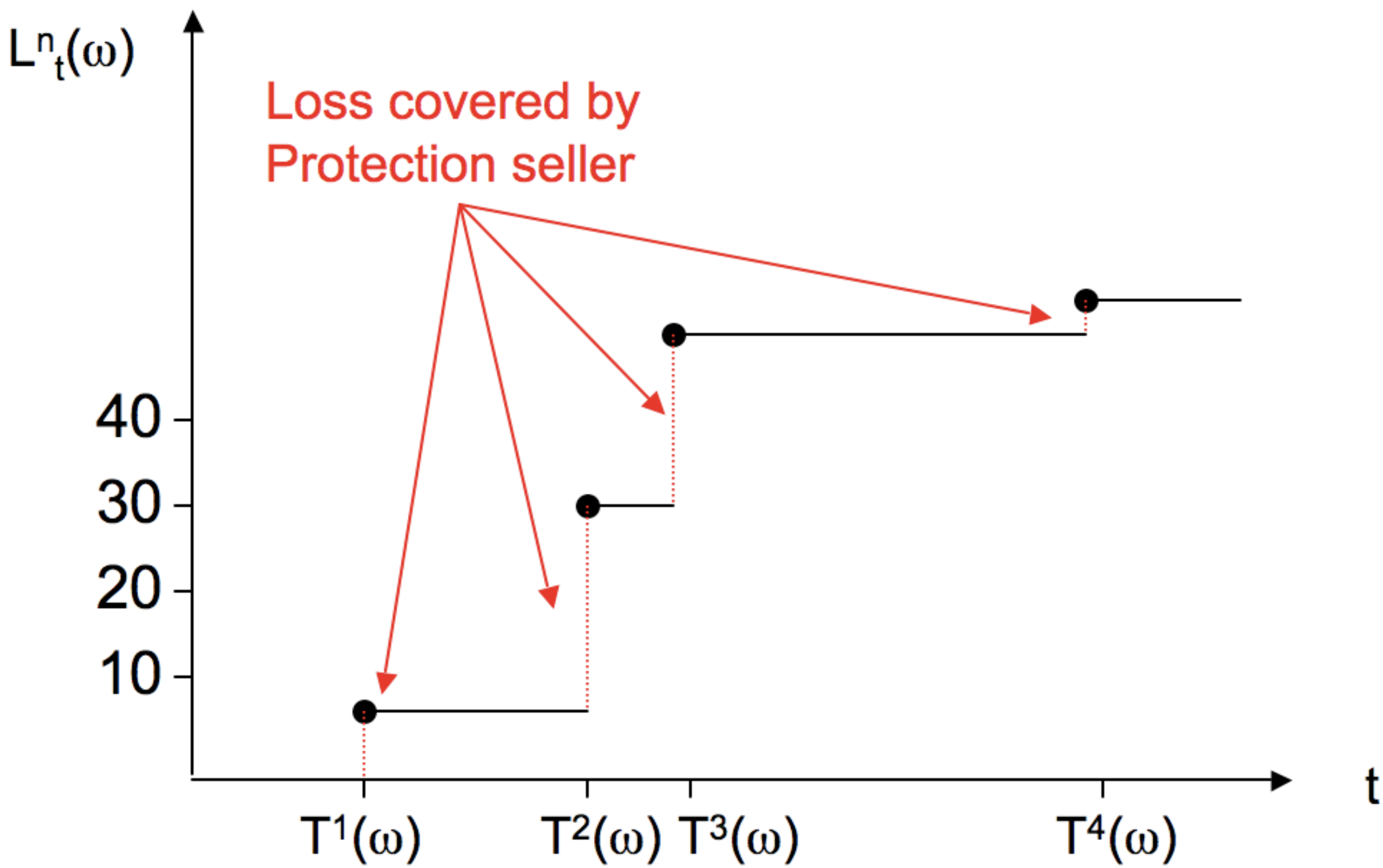
Sample path of default process



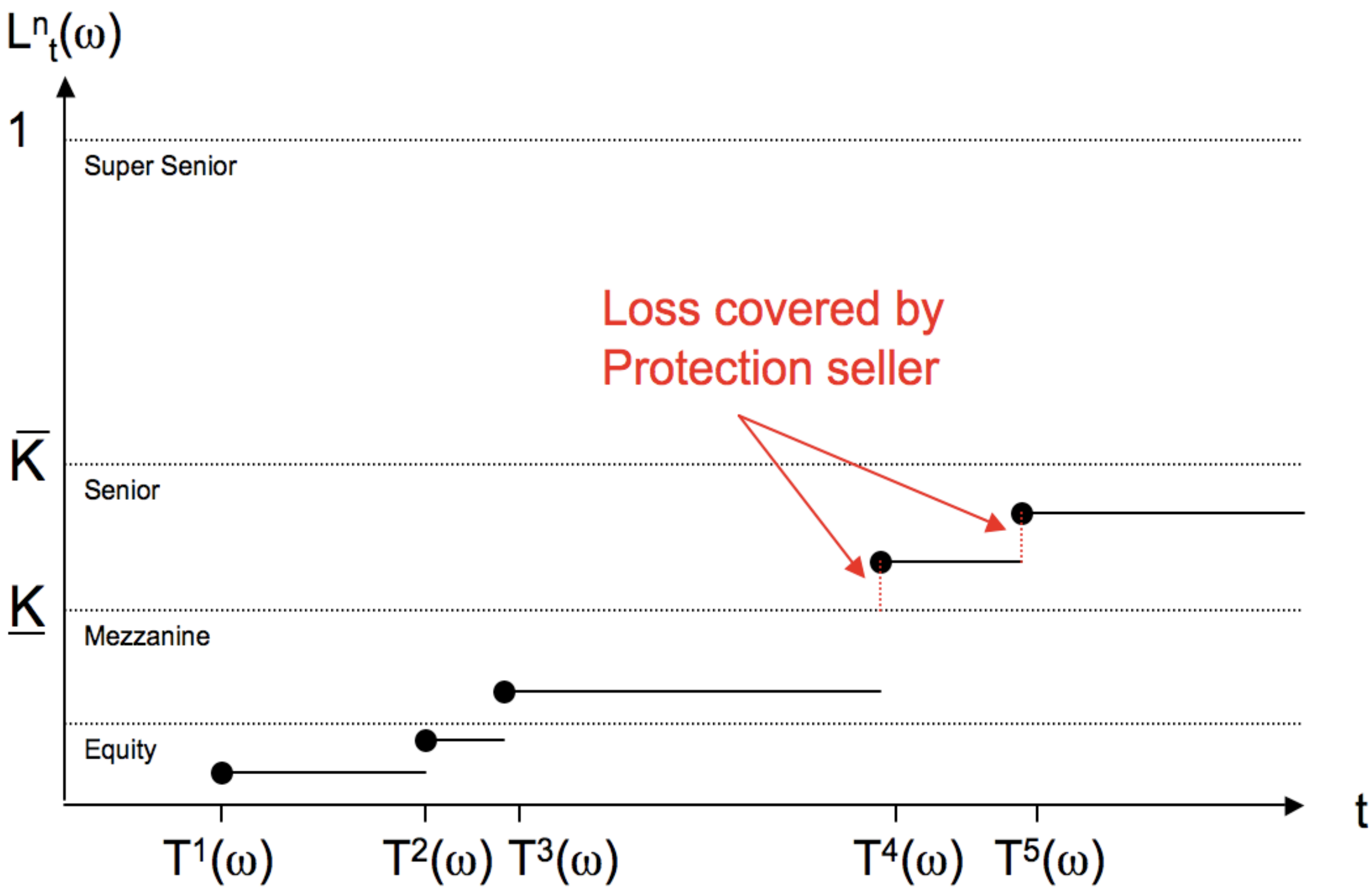
Sample path of loss process



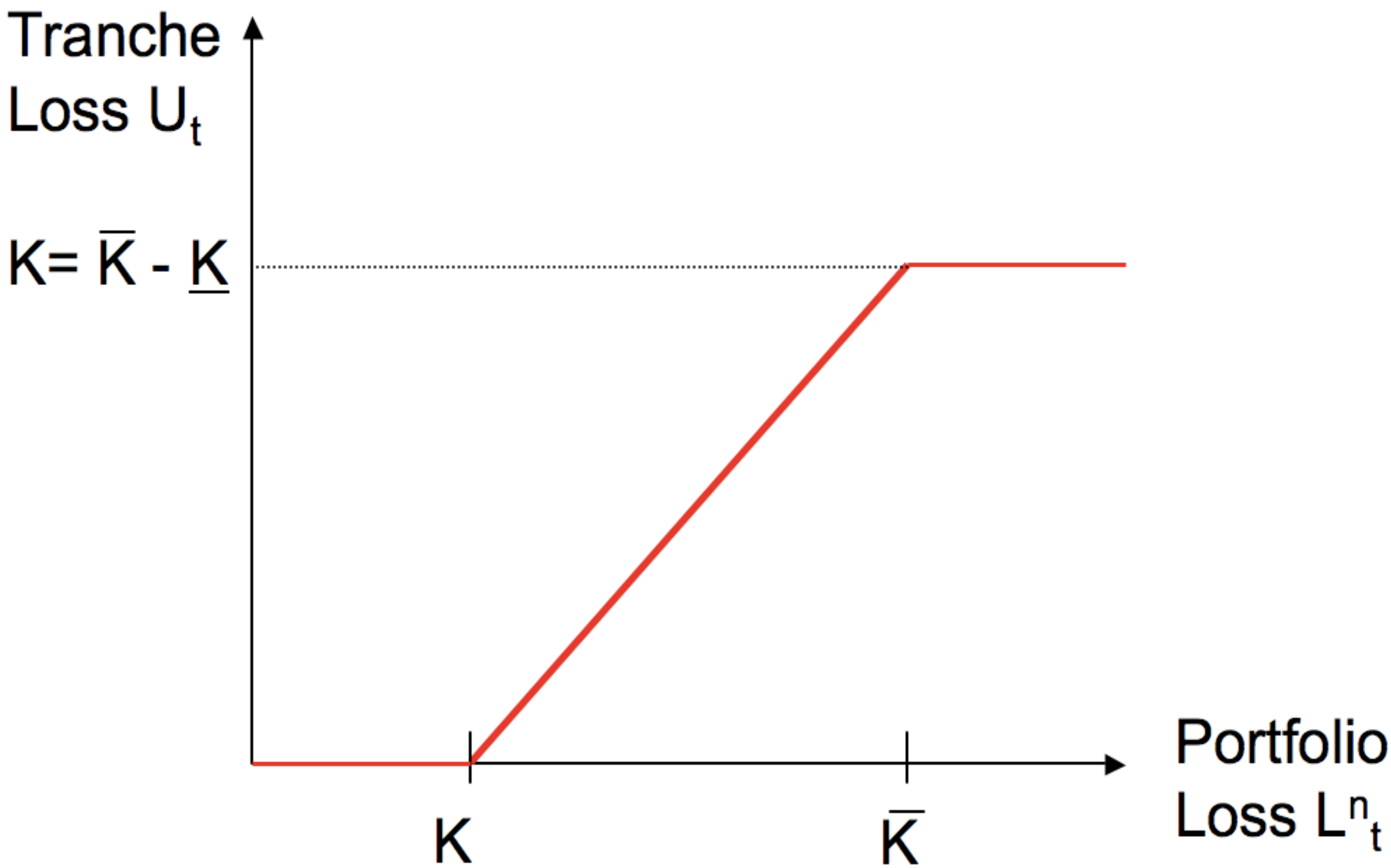
Loss process and index swap



Loss process and tranche swap



Tranche loss = call spread on portfolio loss



Tranche swap

- **Default leg** covers the tranche loss $U_t = (L_t - \underline{K})^+ - (L_t - \overline{K})^+$ up to the maturity T; with a constant interest rate r

$$D_t = e^{-r(T-t)} E_t[U_T] - U_t + r \int_t^T e^{-r(s-t)} E_t[U_s] \, ds$$

- **Premium leg**: stream of payments at (t_m) that are proportional to $K - U_{t_m}$; with S the spread

$$P_t(S) = S \sum_{t_m \geq t} e^{-r(t_m - t)} \big(K - E_t[U_{t_m}]\big)$$

- A top down value of the **tranche spread** at time t is the solution $S = S_t(T)$ to the equation $D_t = P_t(S)$
  - Depends only on the values of call options on L

Top down approach

- The loss process L is specified directly in terms of
  - An **intensity** λ that governs the arrival of events: for small Δt

$$\lambda_t \Delta t \approx P_t\big[\text{event in } (t, t + \Delta t]\big]$$

  - A **distribution** ν that governs the random loss at default ℓ^k
- A portfolio derivative is a contingent claim on L
  - Basic building block is the call option $E_t[(L_s - k)^+]$
- Random thinning allocates L to the portfolio constituents
  - Hedging of single name exposures

Compound Poisson loss process

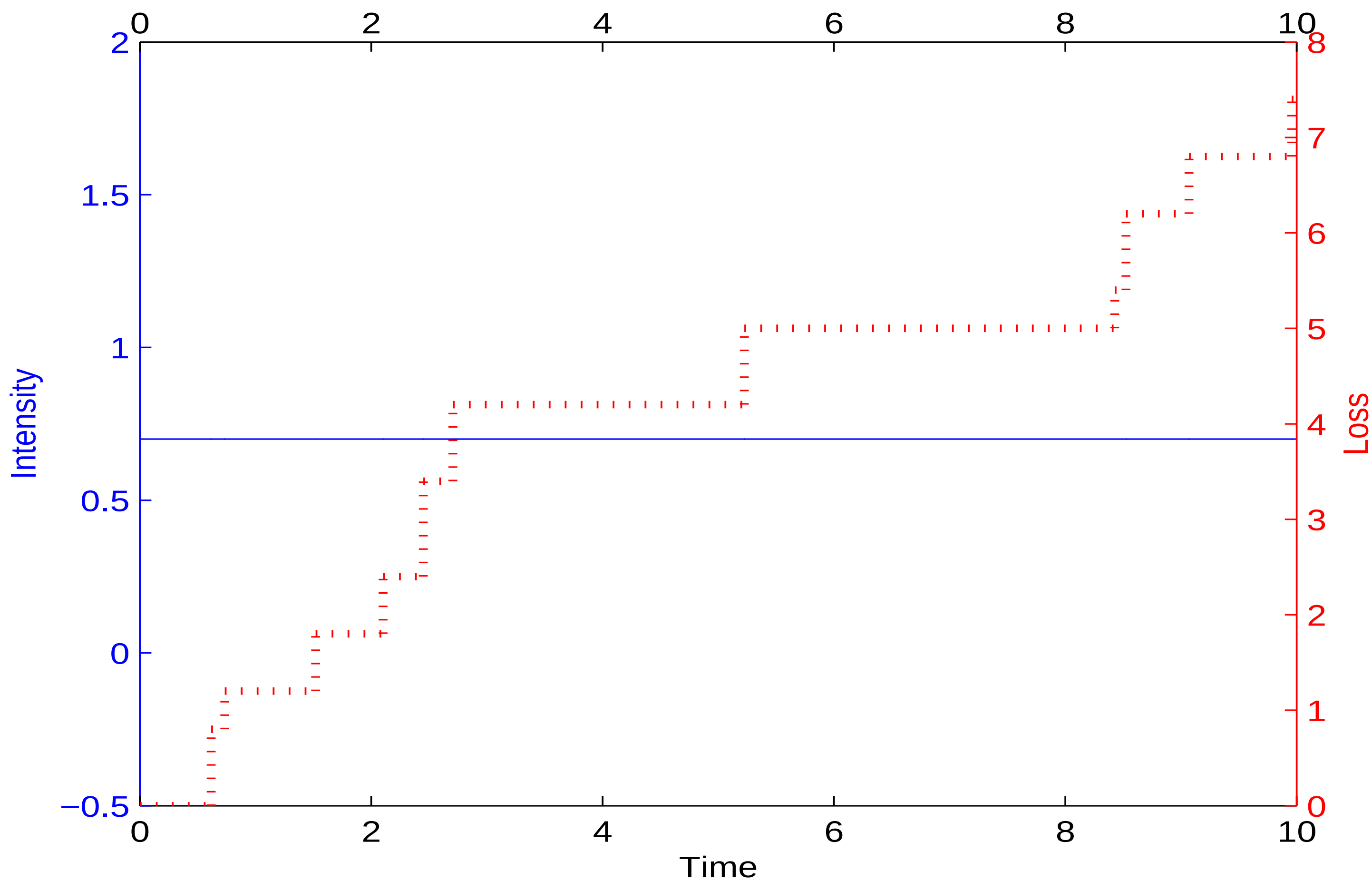
- The intensity λ is given by

$$\lambda_t = c = \text{constant}$$

- Waiting times between events are independent exponential variables with parameter c
- The benchmark top down model for the portfolio loss

Compound Poisson loss process

The loss ℓ^k is uniform on $\{0.4, 0.6, 0.8, 1\}$ and $c = 0.7$



Compound birth loss process

- The intensity λ is given by

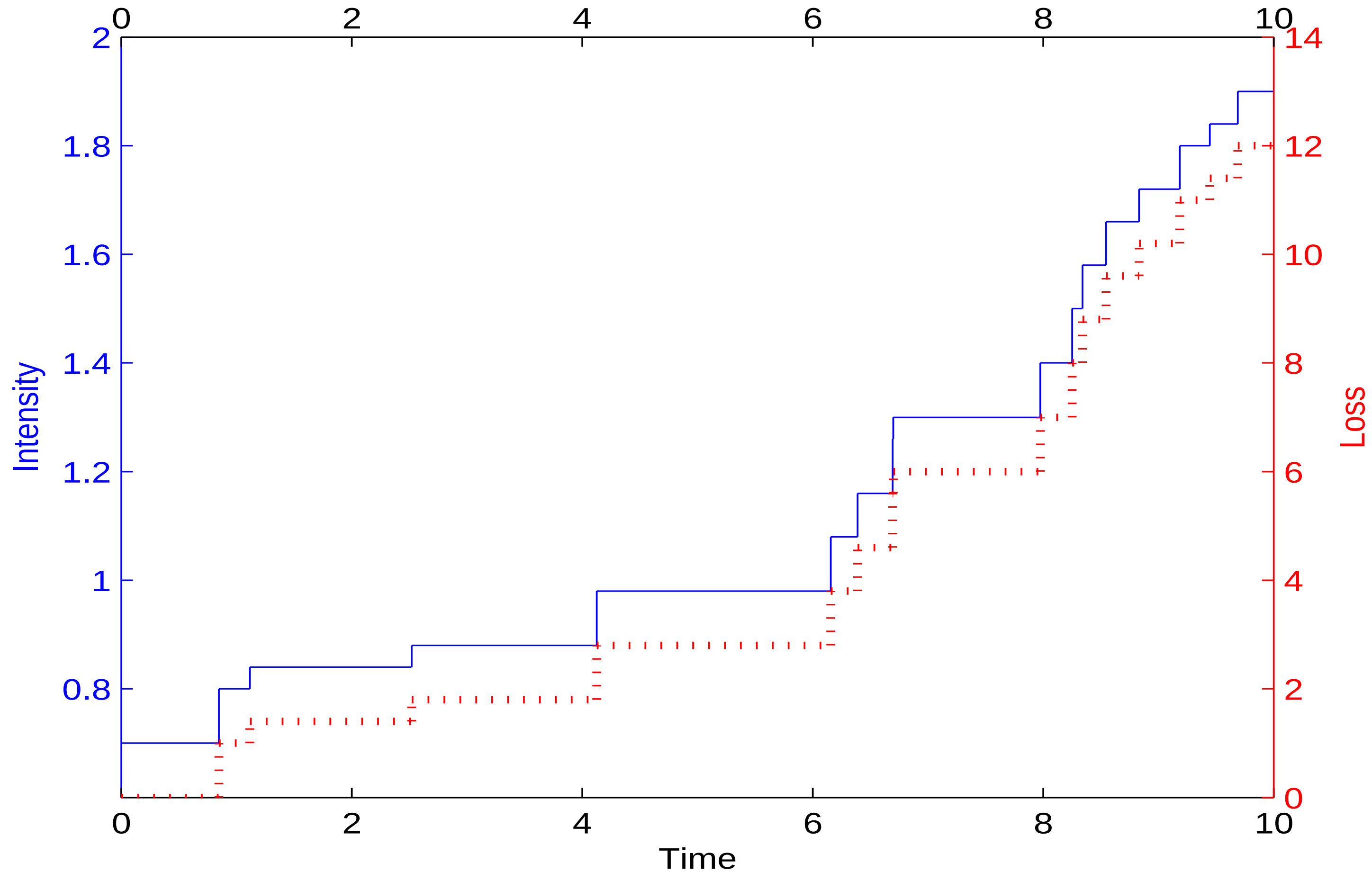
$$\lambda_t = c + \delta L_t$$

- Intensity responds to events; if $\delta = 0$: Poisson process
- **Negative correlation** between default and recovery rates
- Waiting times between events are independent exponentials with (state-dependent) parameter c, $c + \delta\ell^1$, $c + \delta(\ell^1 + \ell^2)$,...
- The intensity has dynamics

$$d\lambda_t = \delta\, dL_t, \quad \lambda_0 = c$$

Compound birth loss process

The loss ℓ^k is uniform on $\{0.4, 0.6, 0.8, 1\}$, $c = 0.7$ and $\delta = 0.1$



Compound Hawkes loss process

- The intensity λ is given by

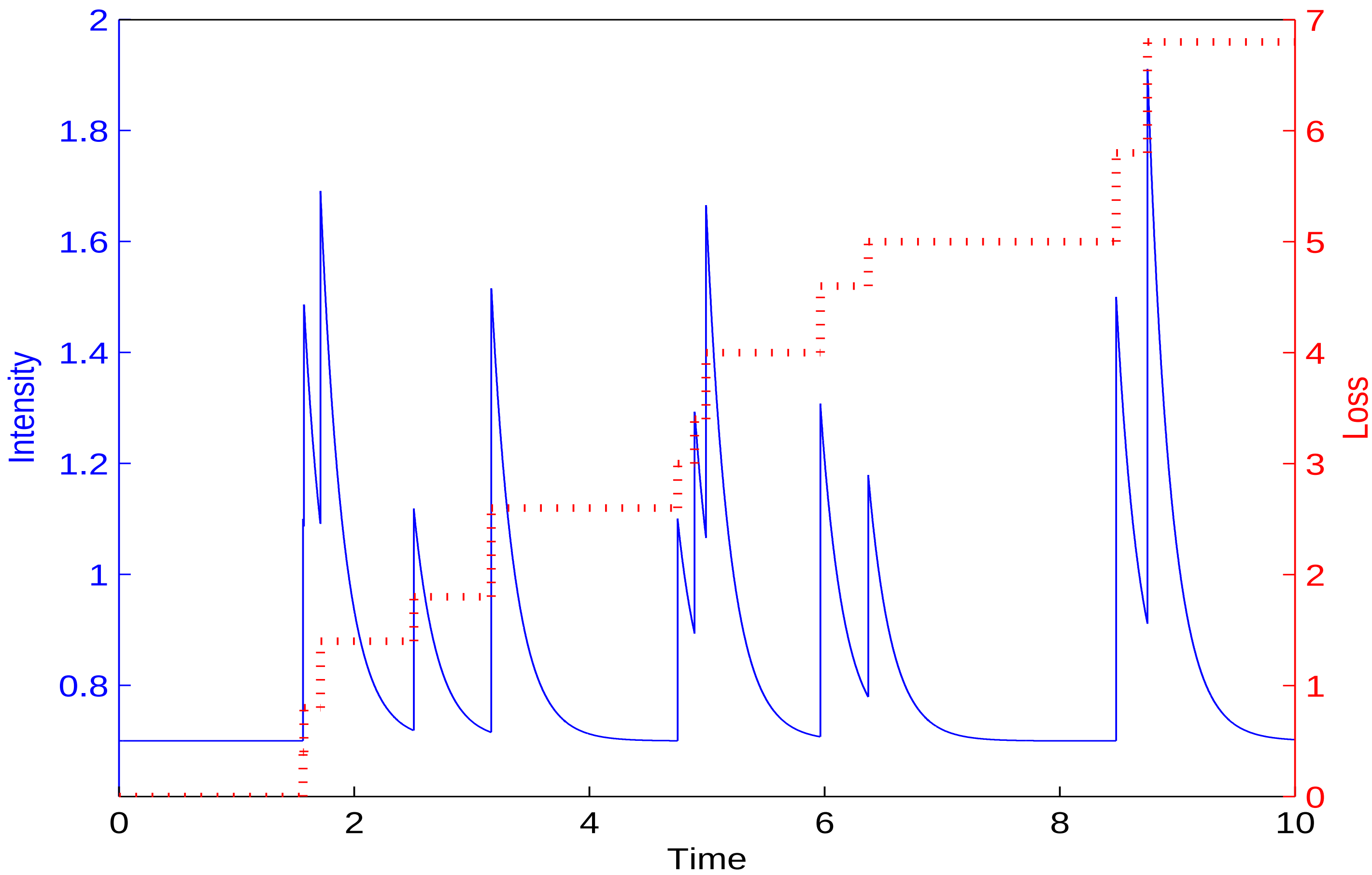
$$\lambda_t = c + (\lambda_0 - c)e^{-\kappa t} + \delta \int_0^t e^{-\kappa(t-s)} dL_s$$

- Intensity responds to events; if $\delta = \kappa = 0$: Poisson process
- Intensity decays after an event; if $\kappa = 0$: birth process
- Negative correlation between default and recovery rates
- The intensity has dynamics

$$d\lambda_t = \kappa(c - \lambda_t)\,dt + \delta\,dL_t, \quad \lambda_0 > 0$$

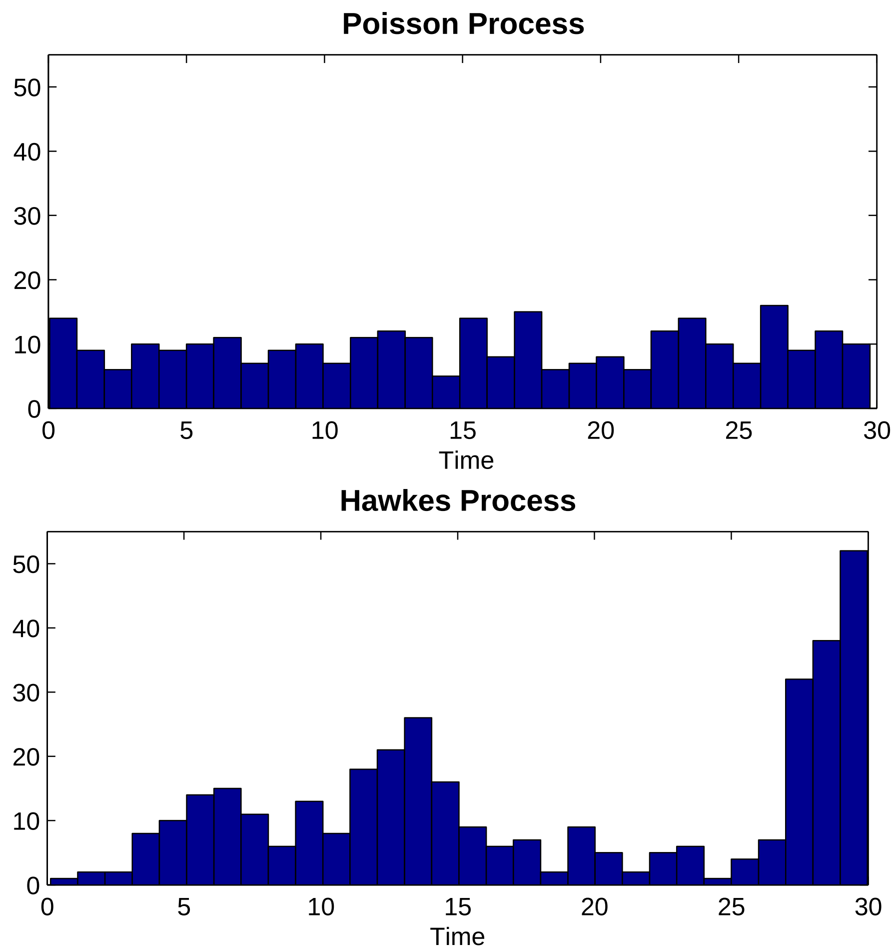
Compound Hawkes loss process

The loss ℓ^k is uniform on $\{0.4, 0.6, 0.8, 1\}$, $c = \lambda_0 = 0.7$, $\delta = 1$, $\kappa = 5$



Hawkes vs. Poisson models

The loss ℓ^k is uniform on $\{0.4, 0.6, 0.8, 1\}$, $c = \lambda_0 = 1$, $\delta = 2$ and $\kappa = 1.5$ for the Hawkes process and $c = 10.57$ for the Poisson process



Basic affine loss process

- The intensity λ is given by

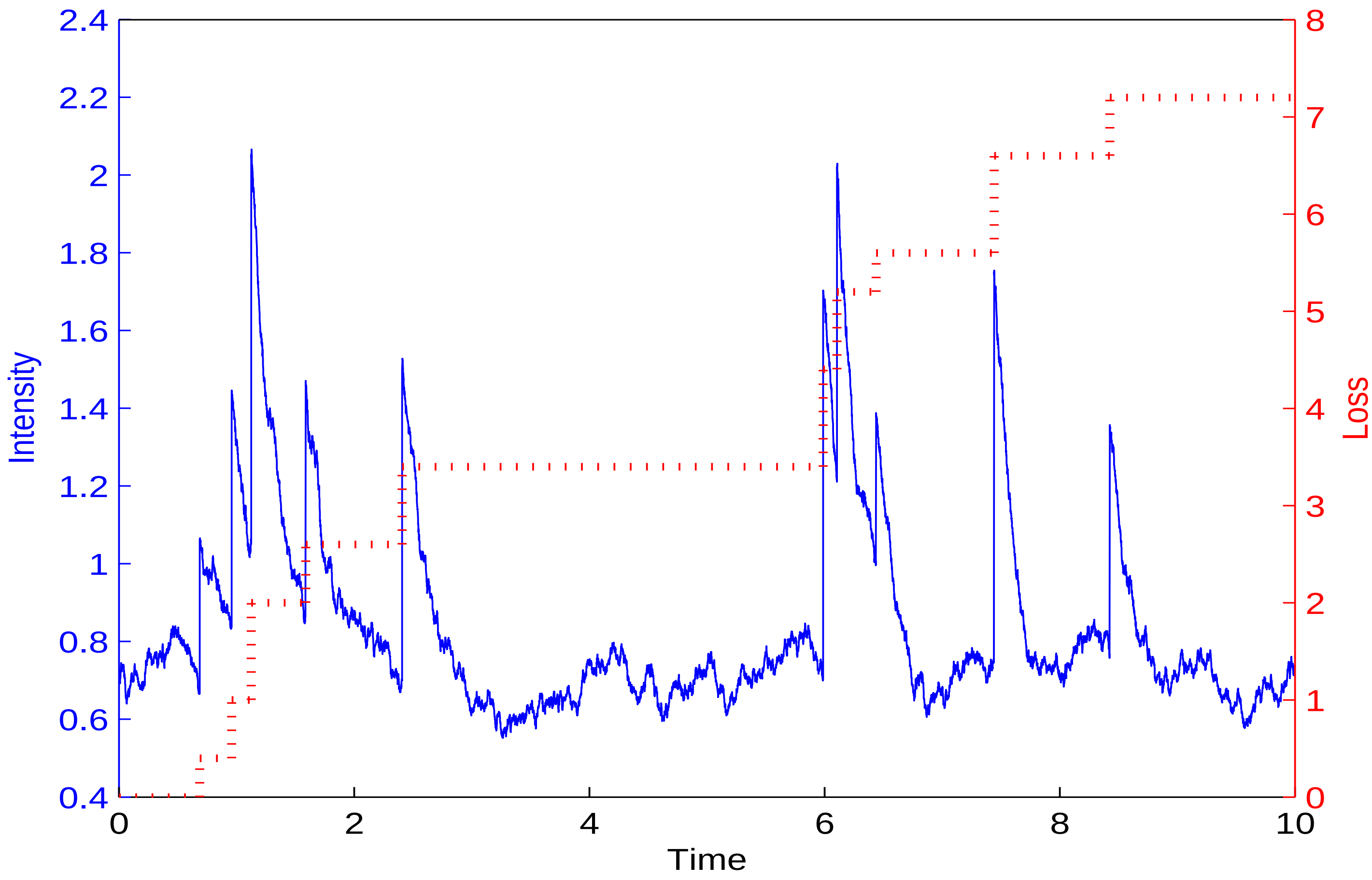
$$\lambda_t = c + (\lambda_0 - c)e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} \sqrt{\lambda_s} dW_s + \delta \int_0^t e^{-\kappa(t-s)} dL_s$$

- Intensity diffuses between events; if $\sigma = 0$: Hawkes process
- Intensity responds to events; if $\delta = \sigma = \kappa = 0$: Poisson process
- Intensity decays after an event; if $\kappa = \sigma = 0$: birth process
- Negative correlation between default and recovery rates
- The intensity has dynamics

$$d\lambda_t = \kappa(c - \lambda_t)\, dt + \sigma \sqrt{\lambda_t} dW_t + \delta\, dL_t, \quad \lambda_0 > 0$$

Basic affine loss process

The loss ℓ^k is uniform on $\{0.4, 0.6, 0.8, 1\}$, $c = \lambda_0 = 0.7$, $\delta = 1$, $\kappa = 5$



Calibration

- We calibrate the basic affine loss process to index and tranche spreads/upfront rates on the CDX.NA.HY.5Y observed on 5/11/2007, which has attachment points $0, 10, 15, 25, 35, 100\%$
- We assume the distribution of loss at default ν is uniform on $\{\ell_1, \ell_2\}$ with $0 < \ell_1 < \ell_2 < 1$, and we set the expected loss at default $E[\ell^k] = \int z d\nu(z) = 0.6$
- The intensity λ is specified by the parameters $\lambda_0, c, \kappa, \sigma$ and δ
- The risk-free rate $r = 0.05$

Calibration

- We fit the parameter vector $\theta = (\lambda_0, c, \kappa, \sigma, \delta, \ell_1)$ by numerically solving the optimization problem

$$\min_{\theta \in \Theta} \sum_i \left(\frac{\text{MarketMid(i)} - \text{Model(i,}\theta\text{)}}{\text{MarketAsk(i)} - \text{MarketBid(i)}} \right)^2$$

subject to $2\kappa c \geq \sigma^2$

where $\Theta = [0,5]^3 \times [0,1] \times [0,5] \times [0.2, 0.6]$ and the sum ranges over the spread/upfront rate data points

- We use adaptive simulated annealing

- We analyze two model specifications:
  - Mod 1 is unrestricted
  - Mod 2 is restricted: the diffusive volatility $\sigma = 0$

Calibration

5Y maturity: market data and fitting results

- Both models fit the data well; the basic affine model does better than the Hawkes model

	MarketBid	MarketAsk	Mod 1	Mod 2
0-10	70.50%	70.75%	71.11%	71.48%
10-15	34.25%	34.50%	32.85%	32.74%
15-25	316.00	319.00	316.80	311.43
25-35	79.00	81.00	81.47	77.34
Index	262.85	263.10	263.46	262.97
MinObj			41.63	60.41
AAPE			1.47%	2.24%

Calibration

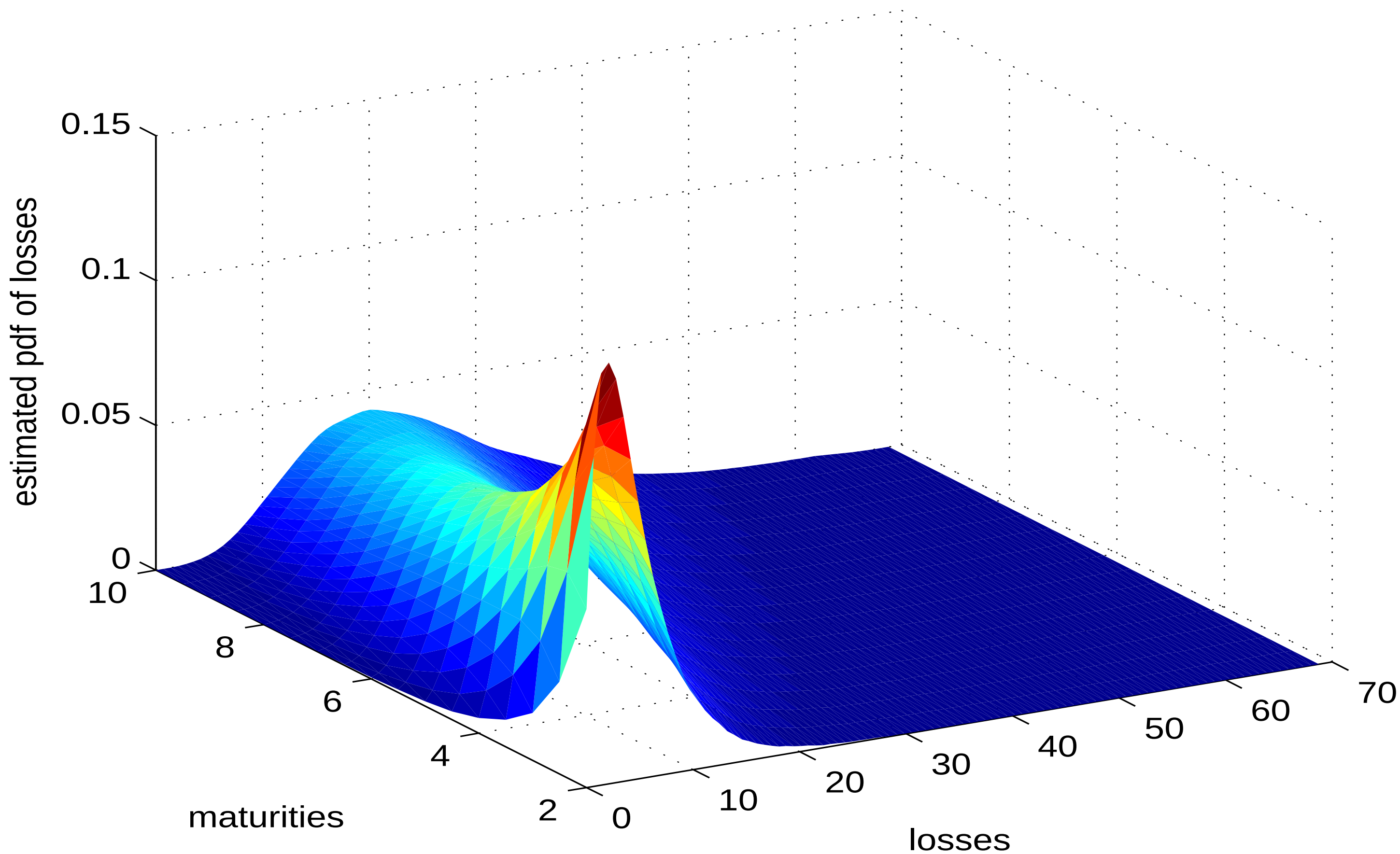
5Y maturity: initial and calibrated parameter values

- We ran several calibrations with different initial values; the values reported below generated the lowest objective function value
- The calibrated parameter values are very similar for the two models

	λ_0	c	κ	σ	δ	ℓ_1
Initial	2.50	2.50	2.50	0.50	2.50	0.40
Mod 1	0.70	1.61	2.62	0.62	2.99	0.24
Mod 2	0.75	1.60	2.58	0.00	2.94	0.24

Calibration

5Y maturity: loss distribution implied by the calibrated basic affine model Mod 1



Calibration

5Y maturity: loss distribution implied by the calibrated model

- The loss distribution for all horizons is captured by one set of parameters $\theta = (\lambda_0, c, \kappa, \sigma, \delta, \ell_1)$, and so is the complete term structure of arbitrage free index and tranche spreads for all attachment points
  - Compare with copula models, which need one correlation parameter per maturity or attachment point
- Further, we have the evolution of the conditional loss distribution for all horizons as the observation date moves forward

Calibration

5Y maturity: parameter stability

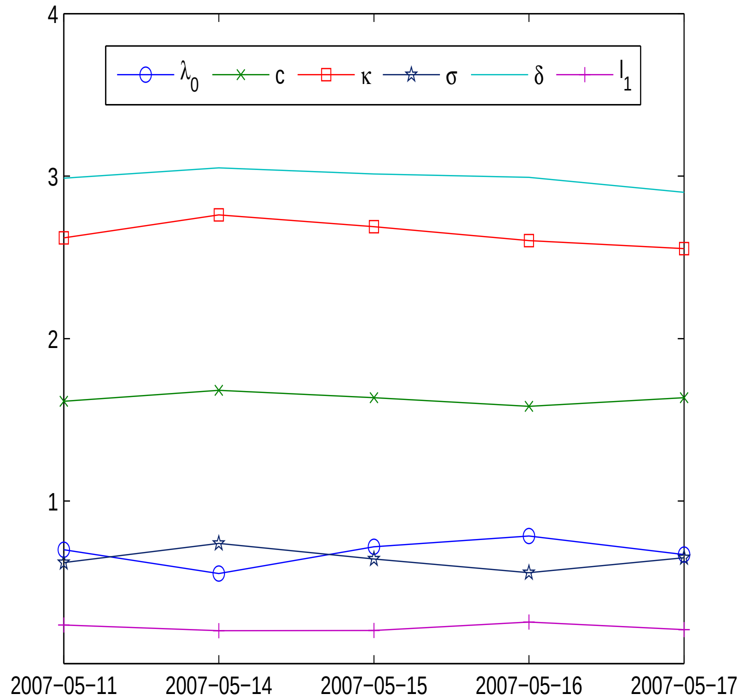
- We re-calibrate at different dates starting with 5/11/07
- The initial values at a date after 5/11 are set to the optimal values from the previous date

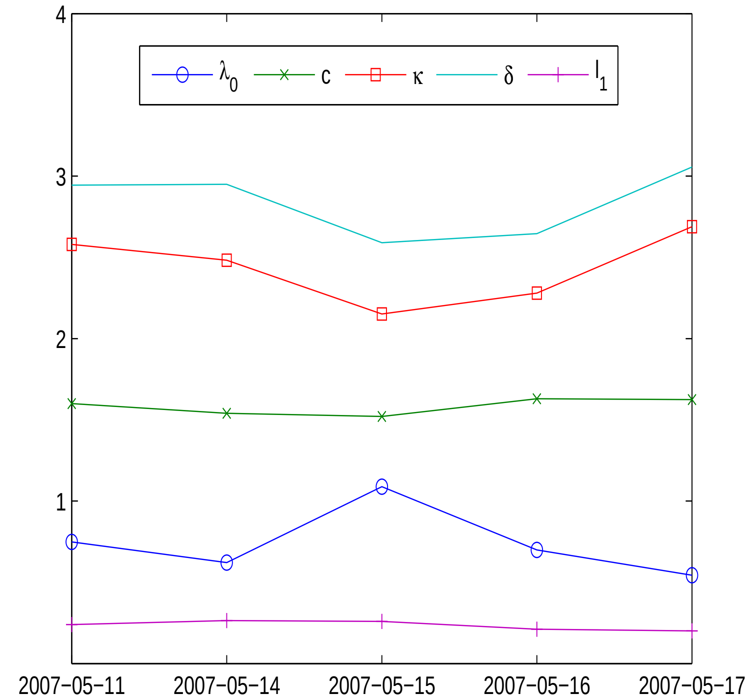
Mod 1	Date	05/11	05/14	05/15	05/16	05/17
	MinObj	41.63	58.66	46.78	46.24	58.24
	AAPE	1.47%	1.70%	1.26%	1.32%	1.35%

Mod 2	Date	05/11	05/14	05/15	05/16	05/17
	MinObj	60.41	73.43	54.29	43.88	65.23
	AAPE	2.24%	2.04%	1.57%	1.45%	1.92%

Calibration

5Y maturity: parameter stability Mod 1 (left) better than Mod 2





Calibration

5 and 7Y maturities: market data and fitting results

		MarketBid	MarketAsk	Mod 1	Mod 2
5Y	0-10	70.50%	70.75%	71.69%	72.00%
	10-15	34.25%	34.50%	33.44%	33.47%
	15-25	316.00	319.00	316.13	309.93
	25-35	79.00	81.00	78.45	72.79
	Index	262.85	263.10	263.94	262.75
7Y	0-10	80.13%	80.38%	81.23%	81.49%
	10-15	55.50%	55.75%	55.17%	55.40%
	15-25	582.00	587.00	580.25	584.63
	25-35	180.00	183.00	206.41	207.98
	Index	307.50	307.75	307.53	308.91
	MinObj			136.12	192.82
	AAPE			2.35%	3.30%

Calibration

5Y maturity: Mod 1 out-of-sample forecast for 7Y

		MarketBid	MarketAsk	5+7Y	5Y
5Y	0-10	70.50%	70.75%	71.69%	71.11%
	10-15	34.25%	34.50%	33.44%	32.85%
	15-25	316.00	319.00	316.13	316.80
	25-35	79.00	81.00	78.45	81.47
	Index	262.85	263.10	263.94	263.46
7Y	0-10	80.13%	80.38%	81.23%	81.79%
	10-15	55.50%	55.75%	55.17%	52.49%
	15-25	582.00	587.00	580.25	531.61
	25-35	180.00	183.00	206.41	186.26
	Index	307.50	307.75	307.53	296.00
	MinObj			136.12	
	AAPE			2.35%	3.04%

Single name hedging

Random thinning

- Models for the default times τ_i of the portfolio constituents are required to estimate the exposure to single name spread changes
- We need to attribute a fraction of the portfolio credit risk to a constituent name
- The portfolio arrival risk is described by the intensity λ so we allocate a fraction Z^i of λ to name i
- The fraction Z^i is parametrized and fit so that we match the single name swap term structure
- The fraction Z^i may follow its own stochastic process, and is called the **thinning** for λ

Single name hedging

Single name default probabilities

- For any model λ, there exists a thinning process Z^i such that

$$P_t\big[t < \tau_i \leq s\big] = \int_t^s E_t\big[Z^i_u \lambda_u\big]\, du$$

- The thinning process satisfies

$$Z^i_t = \lim_{\epsilon \to 0} Z^i_t(\epsilon)$$

where $Z^i_t(\epsilon)$ is the conditional probability at time t that the next defaulter is i, given that a default occurs by time $t + \epsilon$:

$$Z^i_t(\epsilon) = \sum_k \frac{P_t[T^k = \tau_i \text{ and } T^k \leq t + \epsilon]}{P_t[T^k \leq t + \epsilon]}\, 1_{\{T^{k-1} \leq t < T^k\}}$$

Single name hedging

Properties of a thinning process

1. Each Z_t^i is a conditional probability, so $Z_t^i \in [0,1]$ for all i and t
2. Unless the whole portfolio is in default, the Z_t^i must sum to 1 over constituents i
3. Whenever the whole portfolio is in default the Z_t^i must vanish

Single name hedging

Example specification

- Let n be the number of names in the portfolio
- Suppose the thinning processes Z^i are given by

$$Z_t^i = e^{z^i(t)} 1_{\{N_t \le n\}}$$

for deterministic functions that satisfy

$$z^i(t) \le 0, \qquad \sum_{i=1}^{n} e^{z^i(t)} = 1$$

- Therefore

$$P_t\big[t < \tau_i \le s\big] = \int_t^s e^{z^i(u)} E_t\big[\lambda_u 1_{\{N_u \le n\}}\big]\, du$$

which can be calculated for all the models λ discussed above

Single name hedging

Single name CDS

- Given the default probability $F^i(t) = P_0[\tau_i \leq t]$, we can price a CDS referenced on firm i, or other single name derivatives
- If the loss at default is paid at the default, the premium payments occur at times (t_m) and we neglect premium accruals, then the spread on a credit swap with maturity t is given by the formula

$$S^i(t) = \frac{\ell\Big(e^{-rt}F^i(t) + r\int_0^t e^{-rs}F^i(s)ds\Big)}{\sum_{t_m} e^{-rt_m}(1 - F^i(t_m))}$$

where ℓ is the expected loss at default $E[\ell^k]$

Single name hedging

Example specification

- If swaps with maturities $0 = T_0 < T_1 < T_2 < \ldots < T_J$ are available, we can choose the $z^i(u)$ constant between maturities:

$$z^i(u) = z^{ij}, \quad T_{j-1} \le u < T_j, \quad j = 1, \ldots, J$$

- The default probability $F^i(t)$ can then be calculated as a finite sum so the swap spread is easily computed as well

Single name hedging

Calibrating the thinning

- Two-step procedure
  - Fit the parameters of the intensity λ using index and tranche spreads as described above
  - Given the optimal parameter values for λ, separately fit $\{z^i(u)\}_i$ to single name spreads using a standard bootstrapping algorithm
- One-step procedure
  - Fit the intensity λ and thinnings $\{z^i(u)\}_i$ simultaneously from index, tranche and single name spreads

Computational methods

- Semi-analytical characteristic function methods
- Monte Carlo simulation methods
  - Re-scaling algorithm
  - Exact algorithm
  - Approximate thinning algorithm
- Analytical time change methods

Calculating the characteristic function

Compensator

- Consider a non-explosive default counting process N with totally inaccessible event times T^k
- By the Doob-Meyer theorem, there is a unique increasing continuous process A such that $M = N - A$ is a local martingale; this process is called the **compensator** to A
- Specifying the compensator is tantamount to specifying the evolution of N; above we assumed that A has intensity λ:

$$A_t = \int_0^t \lambda_s ds$$

Calculating the characteristic function

Characteristic martingale

- Let $\Psi(v) = 1 - e^{iv}$ and define

$$Z_t = e^{ivN_t + \Psi(v)A_t} \qquad v \in \mathbb{R}$$

- We can show that Z satisfies the exponential equation

$$Z_t = 1 - \Psi(v) \int_0^t Z_{s-} dM_s$$

- It follows that Z is a local martingale

- If for some T^* and constant $c \geq 2$ the expectation of $e^{cA_{T^*}}$ is finite, then for any $T \leq T^*$ the stopped process $Z^T = (Z_{t\wedge T})$ is a uniformly integrable martingale

Calculating the characteristic function

Laplace transform

- For some $s \le T^*$, consider the measure defined by $Z = Z(v)$:

$$\frac{dP^v}{dP}\bigg|_{\mathcal{F}_t} = Z_t^s$$

- Denoting expectation under P^v by E^v, we define the **Laplace transform** of the compensator A under P^v by

$$\mathcal{L}_{t,s}(u, v) = E_t^v\left[e^{-u(A_s - A_t)}\right], \quad u \in \mathbb{C}, \quad v \in \mathbb{R}$$

  – If P^v is a pricing measure and uA is the cumulative interest rate process, then $\mathcal{L}_{t,s}(u, v)$ is the price of a zero coupon bond

- Girsanov's theorem implies that N has P^v-compensator $e^{iv}A$

Calculating the characteristic function

Characteristic function

- **Theorem:** The characteristic function of N is given by

$$E_t\left[e^{iv(N_s - N_t)}\right] = \mathcal{L}_{t,s}(\Psi(v), v)$$

- The calculation of the characteristic function reduces to the calculation of the Laplace transform of A
- Can be generalized to include stochastic interest rates that may depend on N (flight-to-quality)

Calculating the characteristic function

Affine point process

- Consider a default process N with compensator

$$A_t = \int_0^t \lambda_s ds \quad \text{and} \quad \lambda = \Lambda_0 + \Lambda_1 X$$

  for constants Λ_0 and Λ_1 such that $\Lambda(X_t)$ is strictly positive

- The risk factor X follows the affine jump diffusion

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \delta dL_t, \quad X_0 \in \mathbb{R}$$

  where W is an independent Brownian motion and
  - $\mu(x) = K_0 + K_1 x$, where K_0 and K_1 are constants
  - $\sigma^2(x) = H_0 + H_1 x$, where H_0 and H_1 are constants

- Obvious generalization to multiple dimensions, time-dependent coefficients, additional jump terms

Calculating the characteristic function

Affine point process

- We know that under P^v, the compensator of N is $e^{iv}A$, so we can use standard results on AJDs to get

$$\mathcal{L}_{t,s}(u,v) = E_t^v\left[e^{-u\int_t^s \lambda_r dr}\right] = e^{\alpha(t)+\beta(t)X_t}$$

where $\beta(t) = \beta(u,v,t,s)$ and $\alpha(t) = \alpha(u,v,t,s)$ satisfy

$$\partial_t\beta(t) = u\Lambda_1 - K_1\beta(t) - \frac{1}{2}H_1\beta(t)^2 - e^{iv}\Lambda_1(\theta(\delta\beta(t)) - 1)$$
$$\partial_t\alpha(t) = u\Lambda_0 - K_0\beta(t) - \frac{1}{2}H_0\beta(t)^2 - e^{iv}\Lambda_0(\theta(\delta\beta(t)) - 1)$$

with boundary conditions $\beta(s) = \alpha(s) = 0$, and where, for ν the distribution of the loss at default ℓ^k,

$$\theta(v) = \int e^{uz} d\nu(z)$$

Calculating the characteristic function

Example: basic affine loss process

- Consider the special case $H_0 = \Lambda_0 = 0$, $H_1 = \sigma^2$, $\Lambda_1 = 1$, $K_0 = \kappa c$ and $K_1 = -\kappa$ so that

$$d\lambda_t = \kappa(c - \lambda_t)\, dt + \sigma\sqrt{\lambda_t} dW_t + \delta\, dL_t$$

- The ODEs become

$$\partial_t \beta(t) = u + \kappa\beta(t) - \frac{1}{2}\sigma^2\beta(t)^2 - e^{iv}(\theta(\delta\beta(t)) - 1)$$
$$\partial_t \alpha(t) = -c\kappa\beta(t)$$

- The characteristic function of N is

$$E_t[e^{iv(N_s - N_t)}] = \mathcal{L}_{t,s}(\Psi(v), v)$$

Calculating the characteristic function

Example: basic affine loss process

- We can define new coefficient functions a and b such that

$$E_t[e^{iv(N_s - N_t)}] = \exp\big(a(v,t,s) + b(v,t,s)\lambda_t\big)$$

- The coefficient functions $a(t) = a(v,t,s)$ and $b(t) = b(v,t,s)$ solve the ordinary differential equations

$$\partial_t b(t) = 1 + \kappa b(t) - \frac{1}{2}\sigma^2 b(t)^2 - \theta^N(\delta b(t), v)$$
$$\partial_t a(t) = -c\kappa b(t)$$

with boundary conditions $b(s) = a(s) = 0$, and where, for ν the distribution of the loss at default ℓ^k,

$$\theta^N(u,v) = e^{iv}\int e^{uz}\,d\nu(z)$$

Calculating the characteristic function

Example: basic affine loss process

- Similarly, the characteristic function of the loss is

$$E_t[e^{iv(L_s - L_t)}] = \exp\big(a(v,t,s) + b(v,t,s)\lambda_t\big)$$

- The coefficient functions $a(t) = a(v,t,s)$ and $b(t) = b(v,t,s)$ solve the ordinary differential equations

$$\begin{aligned} \partial_t b(t) &= 1 + \kappa b(t) - \frac{1}{2}\sigma^2 b(t)^2 - \theta^L(\delta b(t), v) \\ \partial_t a(t) &= -c\kappa b(t) \end{aligned}$$

with boundary conditions $b(s) = a(s) = 0$, and where, for ν the distribution of the loss at default ℓ^k,

$$\theta^L(u,v) = \int e^{(u+iv)z} d\nu(z)$$

Top down approach: key points

- The portfolio loss process is specified directly in terms of an intensity and a distribution for the loss at an event
  - Capture firm sensitivity to common factors and feedback
  - Capture dependence between default and recovery rates
  - Capture dependence between default and interest rates
- The complete loss surface is described by one set of parameters, and so is the term structure of index and tranche spreads for all attachment points
- Analytical, transform and simulation methods are available to efficiently calculate credit derivative prices
- Single name hedges are generated by thinning the loss process

Notes

- The intensity based top down approach is based on the following papers:
  - Davis & Lo (2001): first application of idea
  - Errais, Giesecke & Goldberg (2006): specification of and valuation for affine loss processes
  - Giesecke & Goldberg (2005): analysis of random thinning
  - Giesecke (2007): characteristic function formula
  - Giesecke & Kim (2007): Monte Carlo simulation methods
  - Ding, Giesecke & Tomecek (2006): specification of and valuation for time-changed birth processes
  - Giesecke & Tomecek (2005): time changes in credit

Notes

- The **intensity based** top down approach is gaining popularity among academics and practitioners; recent references include
  - Arnsdorf & Halperin (2007)
  - Brigo, Pallavicini & Torresetti (2006)
  - Longstaff & Rajan (2006)
  - Lopatin & Misirpashaev (2007)
  - Tavella & Krekel (2006)

References

Arnsdorf, Matthias & Igor Halperin (2007), BSLP: Markovian bivariate spread-loss model for portfolio credit derivatives. Working Paper, Quantitative Research J.P. Morgan.

Brigo, Damiano, Andrea Pallavicini & Roberto Torresetti (2006), Calibration of CDO tranches with the dynamical generalized-Poisson loss model. Working Paper, Banca IMI.

Davis, Mark & Violet Lo (2001), Modeling default correlation in bond portfolios, *in* C.Alexander, ed., 'Mastering Risk Volume 2: Applications', Prentice Hall, pp. 141–151.

Ding, Xiaowei, Kay Giesecke & Pascal Tomecek (2006), Time-changed birth processes and multi-name credit. Working Paper, Stanford University.

Errais, Eymen, Kay Giesecke & Lisa Goldberg (2006), Pricing credit from the top down with affine point processes. Working Paper, Stanford University.

Giesecke, Kay (2007), The correlation neutral measure for counting processes. Working Paper, Stanford University.

Giesecke, Kay & Baeho Kim (2007), 'Estimating tranche spreads by loss process simulation', *Proceedings of the Winter Simulation Conference* .

Giesecke, Kay & Lisa Goldberg (2005), A top down approach to multi-name credit. Working Paper, Stanford University.

Giesecke, Kay & Pascal Tomecek (2005), Dependent events and changes of time. Working Paper, Stanford University.

Longstaff, Francis & Arvind Rajan (2006), An empirical analysis of collateralized debt obligations. Forthcoming, *Journal of Finance*.

Lopatin, Andrei & Timur Misirpashaev (2007), Two-dimensional

Markovian model for dynamics of aggregate credit loss. Working Paper, Numerix.

Tavella, Domingo & Martin Krekel (2006), Pricing nth-to-default credit derivatives in the PDE framework. Working Paper, Octanti Associates.