The Pricing of Interest-Rate Options in an Extended Libor Market Model

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Abstract

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The Libor Market Model (LMM) is one of the most popular interest-rate models used for the pricing of interest-rate options. It's attraction stems from the fact that it is consistent with the Black model holding for interest-rate caps and floors and is very easy to calibrate to the term structure of bond prices and volatilities. However, there is strong evidence that the model in it's simplest form is miss-specified and does not capture market pricing of caps and floors. In this paper, we describe a simple extension of the LMM with the following desirable features:

1. it is easily implemented with the addition of one extra parameter, compared to the standard LMM,
2. all caps and floors have higher prices than in the LMM and implied volatilities exhibit a smile,
3. any cap can be priced using the forward rate and one other cap price.

1 Introduction

The Libor Market Model (LMM) is the most common implementation in practice of the general Heath, Jarrow and Morton (1990) forward rate approach to the valuation of interest-rate derivatives. First proposed by Miltersen, Sandmann and Sondermann (1997) (MSS) and Brace, Gatarek and Musiella (1997) (BGM), the model assumes that the London Interbank Offer Rate (Libor) has a *conditional* probability distribution which is lognormal.[1] The main attraction of the model stems from the fact that it is consistent with the Black (1976) model holding for the pricing of interest-rate caps and floors. This is a model that has long been popular with practicioners and is often referred to as the market model. The model is also very easy to calibrate to the term structure of bond prices and volatilities. However, in spite of its popularity, there is strong evidence that the model in it's simplest form is miss-specified and does not capture market pricing of caps and floors. In this paper, we therefore propose a simple extension of the LMM in an attempt to reflect these market phenomena.

The Black model is a generalized version of the celebrated Black and Scholes (1973) model for the pricing of options on stocks. In this generalized version, there exists a risk-neutral relationship between the value of the option and the *forward price* of the underlying asset, in the sense of Brennan (1979). In context of the pricing of interest-rate options, the model is used in a slightly different form . Here, the risk-neutral valuation relationship is between the option price and the forward *Libor rate*. If the Black model holds for interest-rate caps and floors, the price of any option can be computed simply by knowing the forward Libor rate and its volatility.

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[1] The unconditional distribution is affected by the stochastic drift and is non-lognormal.

In a recent contribution Franke, Huang and Stapleton (2007) show that if certain restrictions are satisfied on the probability density function (PDF) and the pricing kernel, a two-dimensional risk-neutral relationship exists between the value of an option on an asset, the *forward price* of the asset and the value of one other option on the asset. Here, we use a similar idea to show that under certain assumptions, the value of a caplet is related to the forward Libor rate and the value of one other caplet on Libor. [2]

2 Some Previously Reported Results

In this section, we summarise relevant results from four previous papers. These are:

1. "When are options overpriced: The Black-Scholes model and alternative characterisations of the pricing kernel", G. Franke, R.C. Stapleton and M.G. Subrahmanyam, *European Finance Review*, 3, 1, 1999. [FSS]

   The Black-Scholes model assumes that the underlying asset has a lognormal distribution and that the pricing kernel has constant elasticity. FSS shows that given lognormality, if the pricing kernel has declining elasticity, all options are underpriced by the Black-Scholes model. Similar results are reported in

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[2] A number of alternative extensions of the LMM have been suggested in the literature. These also change the assumptions regarding the distribution of Libor rates. Many of these use an assumption of stochastic volatility. It is well known from stock index option models, that stochastic volatlity can generate fat-tailed distributions and implied volatility structures that exhibit smiles. Examples of this approach are to be found in Rebanato (2002) and Brigo and Mercurio (2006)

Bernado and Ledoit (2000). In the present paper, we apply this idea to explain the pricing of interest-rate options.

2. "Two-Dimensional Risk-Neutral Valuation Relationships for the Pricing of Options ", G. Franke, J. Huang and R. C. Stapleton, *Review of Derivatives Research*, 2007. [FHS]

   The Black-Scholes model is an example of a risk-neutral valuation relationship [Brennan (1979)] for the valuation of options. FHS generalizes this idea, introducing a two-dimensional relationship. In this class of models, the price of any option on an asset is related to any two other option prices. In the present paper, we suggest a model in which a similar relationship holds for interest-rate options.

3. "The Black Model and the Pricing of Options on Assets, Futures and Interest Rates" R. C. Stapleton, Teaching note, FIRN 2006.

   The Black (1976) model, a generalized version of the Black-Scholes model, holds given that the underlying asset has a lognormal distribution and that the pricing kernel has constant elasticity. This note shows that these assumptions are not necessary, however. A sufficient condition is that the product of the asset's PDF and the pricing kernel is lognormal. This is the property of the LMM proposed by MSS and BGM. The current paper extends this idea, in the context of interest-rate options, and shows how such options may have higher or lower prices than those predicted by the LMM.

4. " The Libor Market Model: A Recombining Binomial Tree Methodology", S. Derrick, D. J. Stapleton and R. C. Stapleton, working paper, 2005. [DSS]

DSS propose an implementation of the LMM, based on the recombining binomial-tree methodology proposed by Nelson and Ramaswamy (1990) and extended to multi-variate processes by Ho, Stapleton and Subrahmanyam (1995). DSS capture the stochastic drift of the LMM process through a modification of the conditional probabilities in the process. Guan (2007) shows that this implementation of the LMM produces similar prices for European- and Bermudan-style swaptions to those obtained via Monte Carlo simulation while being far more efficient. In the present paper, we use the same idea of adjusting the drift of the process by modifying the conditional probabilities. Our general method incorporates the DSS-LMM and the Black and Karasinski (1991) models as special cases.

3 A No-Arbitrage Economy Setting

Consider a discrete multi-period model in which the price of any security is given by

$$S_t = B_t E_t^Q[\tilde{B}_{t+1} E_{t+1}^Q[\tilde{B}_{t+2}[E_{t+2}^Q[....\tilde{B}_{T-1} E_{T-1}^Q(\tilde{x}_T)]]]], \tag{1}$$

where $\tilde{x}_T$ is the payoff on the security at time T and B_τ is the one-period zero-coupon bond price at time τ. From Pliska (1997), ch 3, (1) holds for some martingale measure Q if and only if there is no-arbitrage. Re-arranging (1) and using the tower property of expectations, S_t can be re-written in the form:

$$S_t = E_t^Q[B_t \tilde{B}_{t+1} \tilde{B}_{t+2} .... \tilde{B}_{T-1} \tilde{x}_T]. \tag{2}$$

Applying (2) to price a European-style contingent claim on x_T, with payoff $c(x_T)$, gives the no-arbitrage price of the claim, $c(S_t)$:

$$c(S_t) = E_t^Q[B_t\tilde{B}_{t+1}\tilde{B}_{t+2}....\tilde{B}_{T-1}c(\tilde{x}_T)], \tag{3}$$

or

$$c(S_t) = E_t^Q[\tilde{\phi}_{t,T}c(\tilde{x}_T)]B_{t,T}, \tag{4}$$

where

$$\phi_{t,T} = \frac{B_t B_{t+1} B_{t+2}....B_{T-1}}{B_{t,T}}. \tag{5}$$

$\phi_{t,T}$ is often called the interest roll-up factor. It is the cash amount that results from investing one dollar in a long bond and discounting it at the one-period zero-coupon bond prices. By construction, we have $E_t^Q(\tilde{\phi}_{t,T}) = 1$. Now let

$$\tilde{\phi}_{t,T}(x_T) \equiv E_t^Q(\tilde{\phi}_{t,T}|x_T)$$

be the projection of $\phi_{t,T}$ on x_T. Then again $E_t^Q[\tilde{\phi}_{t,T}(x_T] = 1$. Also, we have

$$c(S_t) = E_t^Q[\tilde{\phi}_{t,T}(x_T)c(\tilde{x}_T)]B_{t,T}. \tag{6}$$

Equation (6) gives the price of a European-style contingent claim on $\tilde{x}_T$, given only the condition of no-arbitrage. We now develop a more specfic model, under the assumption of (generalized) lognormality.

4 Option Pricing Under Generalized Lognormality

We now make more specific assumptions about the probability distribution of the underlying asset. First, since we will always be pricing a contingent claim at time

t, we simplify the notation, writing $\phi_{t,T}(x_T) \equiv \phi(x_T)$. Also, since we deal only with continuous distributions, we can re-write (6) as

$$c(S_t) = \int c(x_T)\phi(x_T)f(x_T)dx_T B_{t,T}, \tag{7}$$

where $f(x_T)$ is the probabilty density function of x_T under the measure Q. Now, defining $\hat{f}(x_T) \equiv f(x_T)\phi(x_T)$,

$$c(S_t) = \int c(x_T)\hat{f}(x_T)dx_T B_{t,T}. \tag{8}$$

The option price in equation (8) is the expected value of the option payoff under the 'interest-rate adjusted' PDF, $\hat{f}(i_T)$, discounted by the long-bond price, $B_{t,T}$. The probability distribution of x_T under this adjusted mesure is often referred to as the T-period forward measure.

From Stapleton (2006), the Black model holds if and only if $\hat{f}(x_T)$ is the lognormal density function. If $\hat{f}(x_T)$ is lognormal we can write:

$$\hat{f}(x_T) = ag(x_T)e^{q_1 \ln x_T}$$

and this implies that the PDF of x_T is

$$\begin{aligned} f(x_T) &= \hat{f}(x_T)\phi(x_T)^{-1} \\ &= ag(x_T)e^{q_1 \ln x_T - \ln \phi(x_T)}. \end{aligned}$$

which is generalized lognormal.[3]

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[3] The generalized lognormal distribution is defined in FHS as one with a PDF of the form

$$f(x) = ag(x)e^{q_1 \ln x + q_2 k(x)}.$$

In the case of interest-rate options, the underlying 'asset' is an interest rate, usually Libor. For example, an interest rate caplet has a payoff:

$$c(i_T) = \frac{\delta(i_T - k)^+}{1 + \delta i_T},$$

where δ is the period of the underlying Libor loan and k is the strike rate. It follows again from Stapleton (2006), that the Black model holds (with the forward rate substituted for the forward price) if and only if $\hat{f}(i_T)$ is the lognormal density function. If $\hat{f}(i_T)$ is lognormal we can write:

$$\hat{f}(i_T) = ag(i_T)e^{q_1 \ln i_T}$$

and this implies that the PDF of i_T is

$$\begin{aligned} f(i_T) &= \hat{f}(i_T)\phi(i_T)^{-1} \\ &= ag(i_T)e^{q_1 \ln i_T - \ln \phi(i_T)}. \end{aligned} \tag{9}$$

which is generalized lognormal.

The model in (9) reproduces the Black-model prices for caplets and floorlets, i.e. the implied volatilities from the model are flat. To generate a smile (or downward sloping curve) in implied volatilities we need to modify the PDF of i_T under the risk-neutral measure. In the following, we assume that the period length, from t to $t+1$ is δ years and that the risk-neutral measure is the one associated with a δ-length period. We therefore assume now that

$$f(i_T) = ag(i_T)e^{q_1 \ln i_T - \ln \phi(i_T) - \gamma \ln \phi(i_T)}. \tag{10}$$

with $\phi_{t,T}$ given by (5). We refer to this generalized model as the gamma-adjusted model.

We proceed to examine the properties of this 'gamma-adjusted' model for various values of γ. First, if $\gamma = -1$, equation (10) reduces to the lognormal distribution, as assumed in the Black-Karasinski model.[4] Secondly, if $\gamma = 0$, the Black model holds as it does in the LMM, since in this case $\hat{f}(i_T)$ is lognormal. We look now at the general effect of the γ-adjustment on the prices of interest-rate caplets in the case where $\gamma \neq 0$.

First, the price of a caplet at time t is given by

$$cap_t = \int c(i_T)\hat{f}(i_T)di_T B_{t,T}, \tag{11}$$

where $\hat{f}(i_T) = f(i_T)\phi(i_T)$. However, from (11) it is clear that the price depends only on the product $f(i_T)\phi(i_T)$ and not on the individual values of $f(i_T)$ and $\phi(i_T)$. Hence, the price of an option given:

$$\begin{aligned} f(i_T) &= ag(i_T)e^{q_1 \ln i_T + (\gamma-1)\ln\phi(i_T)}, \\ \phi(i_T) &= E(\phi_{i,T}|i_T),\ \phi_{i,T} = \frac{B_t B_{t+1} \ldots B_{T-1}}{B_{t,T}}, \end{aligned}$$

is the same as the price of an option on a lognormal interest rate:

$$\begin{aligned} f(i_T) &= ag(i_T)e^{q_1 \ln i_T}, \\ \text{with}\quad \hat{\phi}(i_T) &= \phi(i_T)^{\gamma-1} = be^{(\gamma-1)\ln\phi(i_T)}. \end{aligned} \tag{12}$$

where the constants a and b gaurantee that $f(i_T)$ and $\hat{\phi}(i_T)$ have unit expectation.

In order to price the option on the interest rate with PDF given by equation (12),

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[4] Black and Karasinski assume a binomial process for the evolution of $\ln i_T$. In the limit, i_T is lognormal in their model.

we can modify the analysis in FSS and FHS. $\hat{\phi}(i_T)$ acts like a pricing kernel[5] with an elasticity which depends on γ. To analyse the pricing of options in the gamma-adjusted model we use the following results:

Lemma 1 *[Franke, Stapleton and Subrahmanyam (1999)]*

[Option Pricing Given a Declining Pricing Kernel]

Given monotonically declining pricing kernels, $\phi_1(x)$ and $\phi_2(x)$, with elasticity $\nu_1(x)$ and $\nu_2(x)$, which give the same forward price, i.e. $F = E[\phi_1(x)x] = E[\phi_2(x)x]$, where F is the forward price of x, then if $\nu_1'(x) = 0$ and $\nu_2'(x) < 0$, then $\phi_1(x)$ and $\phi_2(x)$ intersect twice. Also, all options on x have higher prices under $\phi_2(x)$ than under $\phi_1(x)$.

proof

See Poon and Stapleton (2005)

Lemma 2 *[Option Pricing Given an increasing Pricing Kernel]*

Given monotonically increasing pricing kernels, $\phi_1(x)$ and $\phi_2(x)$, with elasticity $\nu_1(x)$ and $\nu_2(x)$, which give the same forward price, i.e. $F = E[\phi_1(x)x] = E[\phi_2(x)x]$, where F is the forward price of x, then if $\nu_1'(x) = 0$ and $\nu_2'(x) > 0$, then $\phi_1(x)$ and $\phi_2(x)$ intersect twice. Also, all options on x have higher prices under $\phi_2(x)$ than under

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[5] The term pricing kernel usually refers to a variable with unit expectation which reflects the risk aversion of the representative investor. Here, all distributions are under the risk-neutral measure. However, $\hat{\phi}$ is a positive variable, with $E^Q(\hat{\phi}) = 1$ and hence acts as if it were a pricing kernel in this sense.

$\phi_1(x)$.

proof

The existence of the two intersections follows by the same argument as in the proof of Lemma 1. Suppose the intersections are at strike prices k_1 and k_2. Since the pricing kernels are increasing and the elasticity of $\phi_2(x)$ is increasing, $\phi_2(x) > \phi_1(x)$, for $x > k_2$ and for $x < k_1$. Hence all call options with $k > k_2$ and all put options with $k < k_1$ have higher prices under $\phi_2(x)$ than under $\phi_1(x)$. It follows from the argument used to prove Lemma 1 that all options have higher prices under $\phi_2(x)$ than under $\phi_1(x)$.

In the gamma-adjusted model (12), the pricing kernel $\hat{\phi}(i_T)$ is increasing (decreasing) as $\gamma < (>)1$. First, we assume that $\phi(i_T)$ is decreasing in i_T. Then, if $\gamma > 1$ $\hat{\phi}(i_T) = \phi(i_T)^{\gamma-1}$ is also decreasing in i_T. However, if $\gamma < 1$ $\hat{\phi}(i_T) = \phi(i_T)^{\gamma-1}$ is increasing in i_T.

The following result then follows from Lemmas 1 and 2:

Proposition 1 *Cap-Floor Pricing in the Gamma-Adjusted Model*

Let the PDF of the time t Libor interest rate under the δ-period risk-neutral measure be given by the gamma-adjusted function

$$f(i_T) = ag(i_T)e^{q_1 \ln i_T + (\gamma-1)\ln \phi(i_T)},$$

where $\phi_{t,T}$ is given by (5). Then the price of an interest-rate caplet on i_T is higher (lower) than the price given by the Black model when $\gamma < (>)0$.

proof

The price of the option is the same as the price of an option given (12). The elasticity of $\hat{\phi}$ is increasing in i_T. Hence, if $\gamma < [>]0$, $\hat{\phi}$ is increasing [decreasing] in i_T. For $\gamma < [>]0$, by Lemmas 1 and 2, all options have higher [lower] prices than the Black model prices. $\square$

5 Calibration of the Gamma-Adjusted Model

We calibrate the model using a forward induction methodology. As in the case of the Black-Karasinski model, we require the value of each maturity forward contract to be zero, when struck at the forward rate. The value of the T-period long forward rate agreement (FRA) is given by

$$(i_T - k)\phi(i_T)f(i_T)di_T B_{t,T}. \tag{13}$$

Hence we require

$$0 = \int (i_T - f_{t,T})\phi(i_T)f(i_T)di_T B_{t,T}, \tag{14}$$

where $f_{t,T}$ is the forward rate. However, $\phi(i_T)$ depends upon all Libor rates from t to T. Hence, we begin the calibration by determining i_t for $t = 1$, and then proceed forward to determine i_t for $t = 2$ and so forth.

The γ-adjusted model is constructed by modifying the DSS-LMM model described above. The detailed methodology of this latter model is described in the appendix. That model itself is a modification of the Black-Karasinski (BK) model, as illustrated in Table 1, for the case of a three-period example for a one-factor process. In the first panel of the table, we show the conditional probabilities of up-moves in the binomial process for the $T = 1$, $T = 2$ and $T = 3$, forward rates for the BK model. Here, the spot-Libor rates are lognormal and all the conditional probabilities are 0.5. In the second panel of the table, we show the conditional probabilities of up-moves in the case of the DSS-LMM model. The probabilities fix the stochastic drift in the BGM process. As shown in DSS, this model recovers the Black model prices for caplets, in the limit, as the number of binomial steps increases. In the third panel of the table, we show the conditional probabilities of up-moves in the case of the γ-adjusted model, assuming $\gamma = -3$. In this case we use:

$$\begin{aligned}
\hat{q}_{0,1,0} &= q_{0,1,0}\left[\frac{B_{1,2,0}}{B_{0,1,2,0}}\right]^{\gamma} \\
\hat{q}_{0,2,0} &= q_{0,2,0}\left[\frac{B_{1,2,0}B_{2,3,0}}{B_{0,1,2,0}B_{0,2,3,0}}\right]^{\gamma} \\
\hat{q}_{0,3,0} &= q_{0,3,0}\left[\frac{B_{1,2,0}B_{1,2,3,0}B_{1,3,4,0}}{B_{0,1,2,0}B_{0,2,3,0}B_{0,3,4,0}}\right]^{\gamma} \\
\hat{q}_{1,2,0} &= q_{1,2,0}\left[\frac{B_{2,3,0}}{B_{1,2,3,0}}\right]^{\gamma} \\
\hat{q}_{1,3,0} &= q_{1,3,0}\left[\frac{B_{2,3,0}B_{2,3,4,0}}{B_{1,2,3,0}B_{1,3,4,0}}\right]^{\gamma} \\
\hat{q}_{2,3,0} &= q_{2,3,0}\left[\frac{B_{3,4,0}}{B_{2,3,4,0}}\right]^{\gamma}.
\end{aligned}$$

Here, we show only the probability adjustment at node 0. Similar calculations are used at each node throughout the tree. Constructing the conditional probabilities of up-moves in this way guarantees that the unconditional probabilities of the process conform to the required form.

In Table 2, we illustrate the Libor rates resulting from calibrating the $\gamma = -3$ model to the given forward rates (flat 6%). Since the up-move probabilities are higher in this model, the rates are generally lower.

6 Some Preliminary Results from the Model

In this section, we report some preliminary results from the γ-adjusted model. We calibrate the model to recent market data on caplet implied volatilities and forward rates. Figure 1 plots the implied volatilities for caplets reported by Bloomberg, for May 25, 2006. On that date, forward rates were around 5.5% and flat out to three years. At-the-money caplets were selling in the range of 14.2%-15.2%, with volatilities increasing with maturity. In our simulations, we assume Libor volatilities of 14% out to 7 years, and we assume quarterly time steps.

Figure 2 shows the probability density function resulting from different γ adjustment models. The middle curve is the one for $\gamma = 0$. This PDF is consistent with the LMM holding. With $\gamma = -10$, the resulting PDF has fatter tails and a lower peak. The PDF results from calibrating the model to the assumed forward rate of 6%. With $\gamma = 10$, the resulting PDF has thinner tails and a higher peak than in the LMM case with $\gamma = 0$.

Figure 3 shows the prices of caplets at various strike rates. The options are priced by discounting at stochastic rates through the trees. The prices converge at low strike rates, since they are deep in the money, and at high strike rates, since they have zero value. The higher option prices at medium strike rates are produced by the $\gamma = -10$ model, while the middle curve shows the Black prices produced using $\gamma = 0$.

In Figure 4 we report the same prices in terms of implied volatilities found by inverting the Black model. Again, $\gamma = -10$ produces the higher prices, while $\gamma = 0$ produces a flat implied volatility curve, consistent with the Black model holding.

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7 Appendix: The HSS Recombining Node Methodology and the Libor Market Model

Ho, Stapleton and Subrahmanyam (1995) [HSS] suggest a general methodology for creating a recombining multi-variate binomial tree to approximate a multi-variate lognormal process. An adaptation of this methodology has been used by Peterson, Stapleton and Subrahmanyam (2003) [PSS] to build a two-factor spot rate model of the term-structure. In this section we show how a similar application can be made in the case of the LMM. There are some differences in this case however. First, we will assume in this version of the LMM that the stochastic factors driving the term structure are independent log-Brownian motions. Hence, there is no mean reversion or correlation in the factors. The covariances between forward rates are generated by factor loadings on the factors.

We assume a given term structure of forward Libors, $f_{0,T},\ T = 0, 1, ..., N$, where N is the terminal date of the model and a corresponding set of caplet volatilities, $\sigma(T),\ T = 1, ..., N$. From these caplet volatilities we derive a set on N forward volatilities, using the bootstrap method, assuming that the forward rate volatilities depend on the forward maturity, T, and not on time t. We denote these as σ_T. As in Hull and White (2000), we now assume these volatilities are generated by a two-factor model, with factor loadings: $\beta_{1,T}$, $\beta_{2,T}$, where

$$\beta_{i,T} = \alpha_{i,T}\sigma_{T+1},\ T = 0, 1, ..., N - 1.$$

In the two-factor model, the exogenous factors, $\alpha_{i,T}$ are restricted by the relation

$$\alpha_{1,T}^2 + \alpha_{2,T}^2 = 1,$$

and $0 < \alpha_{1,T} < 1$. The covariance between any two forward rates is then given by

$$\sigma_{\tau,T} = \beta_{1,\tau}\beta_{1,T} + \beta_{2,\tau}\beta_{2,T}. \tag{15}$$

For convenience, we denote the part of this covariance generated by factor i as $\sigma_{\tau,T}(i) = \beta_{i,\tau}\beta_{i,T}$.

In HSS, a binomial process is used to approximate a stock-price process with a given volatility and drift structure. Here we use a similar methodology, applying it to each of the factors which generate the forward rates. We first split the drift of the forward rate into two parts: a drift which is due to the volatility of the first factor and a drift which is due to the volatility of the second factor. Since $\sigma_{\tau,T} = \sigma_{\tau,T}(1) + \sigma_{\tau,T}(2)$, the drift can be written

$$\begin{aligned} E\left[f_{t+1,t+T}\right] - f_{t,t+T} &= f_{t,t+T}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\left[\sigma_{T-1,T-1}(1) + \sigma_{T-1,T-1}(2)\right] \\ &+ f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\left[\sigma_{T-2,T-1}(1) + \sigma_{T-2,T-1}(2)\right] \\ &+ \cdots \end{aligned}$$

and therefore

$$\begin{aligned} E\left[f_{t+1,t+T}\right] - f_{t,t+T} &= f_{t,t+T}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1}(1) + f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1}(1) + \cdots \\ &+ f_{t,t+T}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1}(2) + f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1}(2) + \cdots \end{aligned}$$

A straightforward implementation of the LMM would build a non-recombining binomial tree for each factor, using the factor loadings $\beta_{i,T}$ and the drifts above. The resulting bivariate tree would have an exploding number of nodes, but could be used to value interest-rate options using Monte-Carlo analysis. As an alternative, we build a recombining binomial tree for each factor using the techniques in HSS and Nelson and Ramaswamy (1990) and then capture the required drift by using state dependent conditional probabilities.

First, we denote the proportionate up and down movements in the log-binomial process due to factor $i = 1, 2$ as $u(T)(i)$ and $d(T)(i)$ respectively, for the forward rate with maturity T, where the up and down moves of the processes depend only on the maturity of the forward, T, and not on the time t. The T-period forward rate at time t, in state r, s, [after r down-moves in factor 1 and s down-moves in factor 2] is given by

$$f_{t,t+T,r,s} = f_{0,t+T}[u_T(1)]^{t-r}[d_T(1)]^r[u_T(2)]^{t-s}[d_T(2)]^s \tag{16}$$

where

$$d_T(i) = \frac{2}{1 + e^{2\beta_{i,T}\sqrt{\delta}}}$$
$$u_T(i) = 2 - d_T(i),$$

for

$$t = 1, 2, ..., N$$
$$T = 0, 1, ..., N - t.$$

Choosing the proportionate up and down moves, u and d, in this way ensures that the annualised volatility of the Tth forward is exactly σ_T, in the case where all probabilities in the tree are[6] 0.5. Also the tree of forward rates is recombining with $(t + 1)^2$ nodes at time t.

---

[6] If all the conditional probabilities are 0.5, the binomial process has a volatility exactly equal to σ_T. However, when corrected to account for the drift, the probabilities will diverge from 0.5 and the volatility of the binomial process will understate that of the true process.

In Figure 1 we illustrate the recombining tree for the two-factor case. In this example, the volatility of the first factor declines over time, while the second factor has constant volatility. Note that, as in HSS, the tree is forced to recombine, in spite of the fact that the volatility of the first factor declines over time. The transition probabilities depend both on volatility structure and the drifts of the factors. For example, the conditional probability at time 1 of the two-period forward moving up at node (0,0), due to factor 1, is denoted as $q_{1,2,0,0}(1)$. In general for factor i, at time t in state (r, s) this is denoted as $q_{t,t+T,r,s}(i)$.

In order to fix the conditional probabilities in the binomial process, using the HSS methodology, we need to determine the drift of the logarithm of the forward rates. In HSS Theorem 1, it is shown that the binomial process converges to the required lognormal process if the conditional probabilities are chosen using the logarithmic regression of prices on previous prices. Here we follow the same logic, based on the logarithm of forward rates. Dividing the drift equation by $f_{t,t+T}$,

$$\begin{aligned}\frac{E\left[f_{t+1,t+T}\right]-f_{t,t+T}}{f_{t,t+T}} &= \frac{E\left[\Delta f_{t,t+T}\right]}{f_{t,t+T}} \\ &= \frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1}+f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1}+\cdots\end{aligned}$$

Then, for small changes, using Ito's lemma

$$\begin{aligned}\frac{E\left[d\ln f_{t,t+T}\right]}{f_{t,t+T}} &= \frac{E\left[df_{t,t+T}\right]}{f_{t,t+T}}-\frac{\sigma_{T,T}}{2} \\ &= \frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1}+f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1}+\cdots-\frac{\sigma_{T,T}}{2},\end{aligned}$$

where $\sigma_{T,T}=\sigma_T^2$ is the variance of the Tth forward rate.

In the HSS binomial tree, the conditional probability of an up move in the T-maturity forward due to factor i, at time t at node r, s is

$$\begin{aligned} q_{t,t+T,r,s}(i) &= [m_{t,t+T,r,s}(i) + (t-r)\ln u_{T+1}(i) + r\ln d_{T+1}(i) - (t-r)\ln u_T(i) - r\ln d_T(i) \\ &- \ln d_T(i)]/[\ln u_T(i) - \ln d_T(i)], \end{aligned} \tag{17}$$

where $m_{t,t+T,r,s}(i)$ is the annualised logarithmic drift of factor i, and δ is the length of the period t to $t+1$. The drift of the forward rate can now be allocated between the two factors, choosing

$$m_{t,t+T,r,s}(i) = \frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T-1,T-1}(i) + f_{t,t+T-1}\frac{\delta f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-2,T-1}(i) + \cdots - \frac{\sigma_{T,T}(i)}{2},$$

where $\sigma_{T,T}(1) + \sigma_{T,T}(2) = \sigma_{T,T}$.

This completes the description of the HSS application for the case where there is just one up or down move, for each factor, over each period of length δ. However, the general HSS methodology allows for an increase in the binomial density, n, (the number of up or down moves per period). When $n \neq 1$ the above formula generalises as in HSS. Also, as n increases, the variance of the forward rate process converges to the given volatilities. This follows from HSS, Theorem 1.

Table 1: Conditional Probabilities: Three-Period Model

Black-Karasinski Model ($\gamma = 1$)	$q_{0,1}$	0.50				
	$q_{0,2}$	0.50	$q_{1,2,0}$	0.50		
			$q_{1,2,1}$	0.50		
	$q_{0,3}$	0.50	$q_{1,3,0}$	0.50	$q_{2,3,0}$	0.50
			$q_{1,3,1}$	0.50	$q_{2,3,1}$	0.50
					$q_{2,3,2}$	0.50
Libor Market Model ($\gamma = 0$)	$q_{0,1}$	0.5054				
	$q_{0,2}$	0.5112	$q_{1,2,0}$	0.5066		
			$q_{1,2,1}$	0.5044		
	$q_{0,3}$	0.5173	$q_{1,3,0}$	0.5137	$q_{2,3,0}$	0.5081
			$q_{1,3,1}$	0.5093	$q_{2,3,1}$	0.5055
					$q_{2,3,2}$	0.5037
$\gamma = -3$ Model	$q_{0,1}$	0.5229				
	$q_{0,2}$	0.5477	$q_{1,2,0}$	0.5279		
			$q_{1,2,1}$	0.5189		
	$q_{0,3}$	0.5746	$q_{1,3,0}$	0.5587	$q_{2,3,0}$	0.5343
			$q_{1,3,1}$	0.5539	$q_{2,3,1}$	0.5234
					$q_{2,3,2}$	0.5158

1. Assumes volatility is flat 20%, and the forward curve is flat 6%. The period length, $\delta = 1$.
2. $q_{t,T,r}$ is the conditional probability at t and node r of an up-move in the T-period forward rate. r denotes the number of down moves in the binomial process.
3. The LMM probabilities are from Derrick, Stapleton and Stapleton (2005).

Table 2: Calibrated Rates: Three-Period Model

	node	$f_{0,0}$	$f_{1,1}$	$f_{2,2}$	$f_{3,3}$
BK model ($\gamma = 1$)	0	0.06	0.0720	0.0866	0.1044
	1		0.0483	0.0581	0.0700
	2			0.0384	0.0459
	3				0.0314
	node	$f_{0,0}$	$f_{1,1}$	$f_{2,2}$	$f_{3,3}$
LMM model ($\gamma = 0$)	0	0.06	0.0718	0.0860	0.1030
	1		0.0482	0.0577	0.0690
	2			0.0387	0.0463
	3				0.0310
	node	$f_{0,0}$	$f_{1,1}$	$f_{2,2}$	$f_{3,3}$
$\gamma = -3$ model	0	0.06	0.0714	0.0843	0.0983
	1		0.0478	0.0565	0.0659
	2			0.0379	0.0442
	3				0.0296

1. Assumes volatility is flat 20%, the forward curve is flat 6% and period length, $\delta = 1$.

2. Conditional probabilities in the process are those shown in Table 1.

3. The stochastic rates satisfy the integral equations:

$$
\begin{aligned}
FRA_{0,1} = 0 &= E_0[(i_1 - 0.06)B_{1,2}]B_{0,1} \\
FRA_{0,2} = 0 &= E_0[(i_2 - 0.06)B_{1,2}B_{2,3}]B_{0,2} \\
FRA_{0,3} = 0 &= E_0[(i_3 - 0.06)B_{1,2}B_{2,3}B_{3,4}]B_{0,3}
\end{aligned}
$$

Figure 1: Caplet Implied Vols, 2006 May 25 (source Bloomberg)

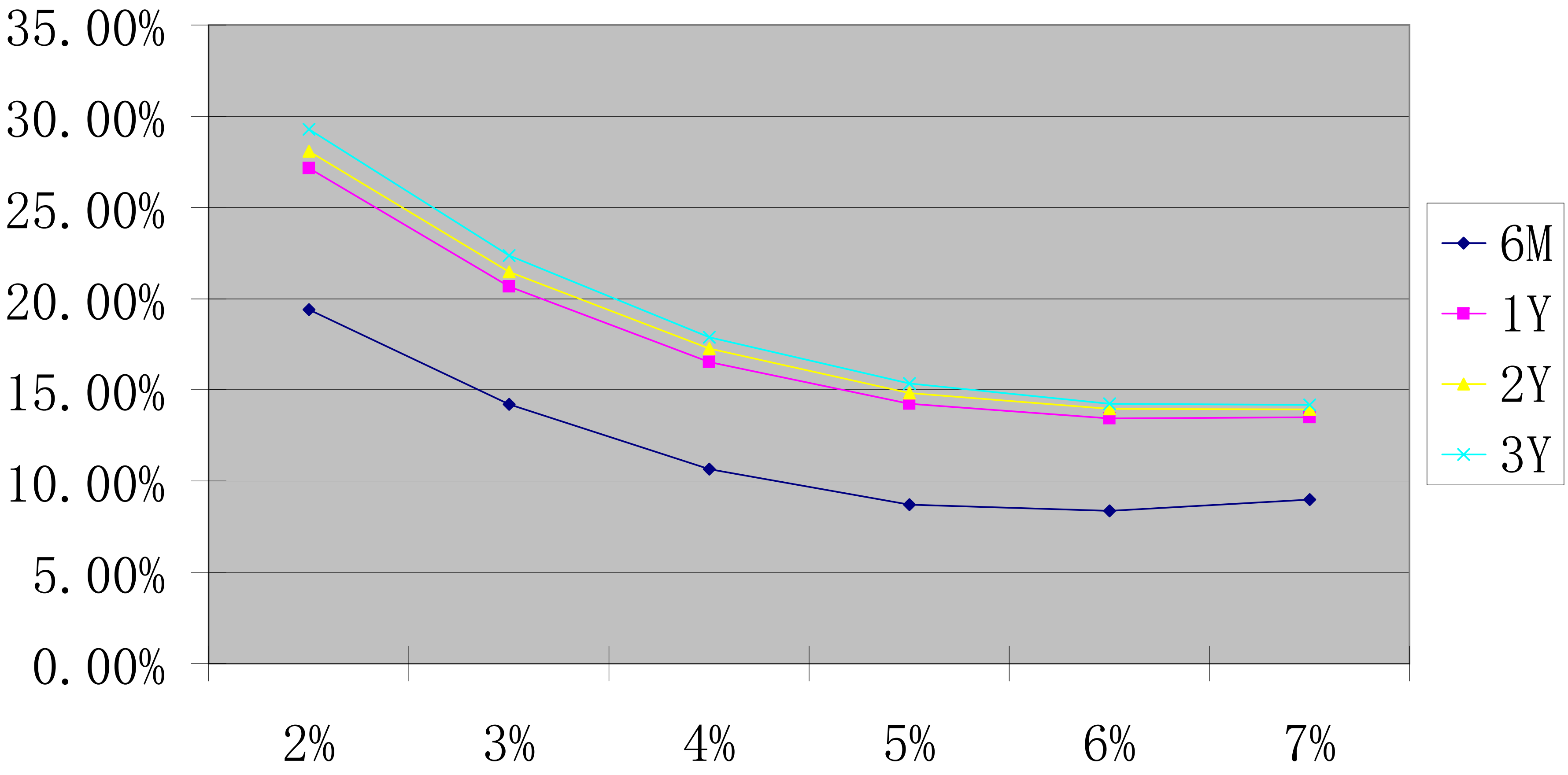

Figure 2: PDF (Effect of Gamma)

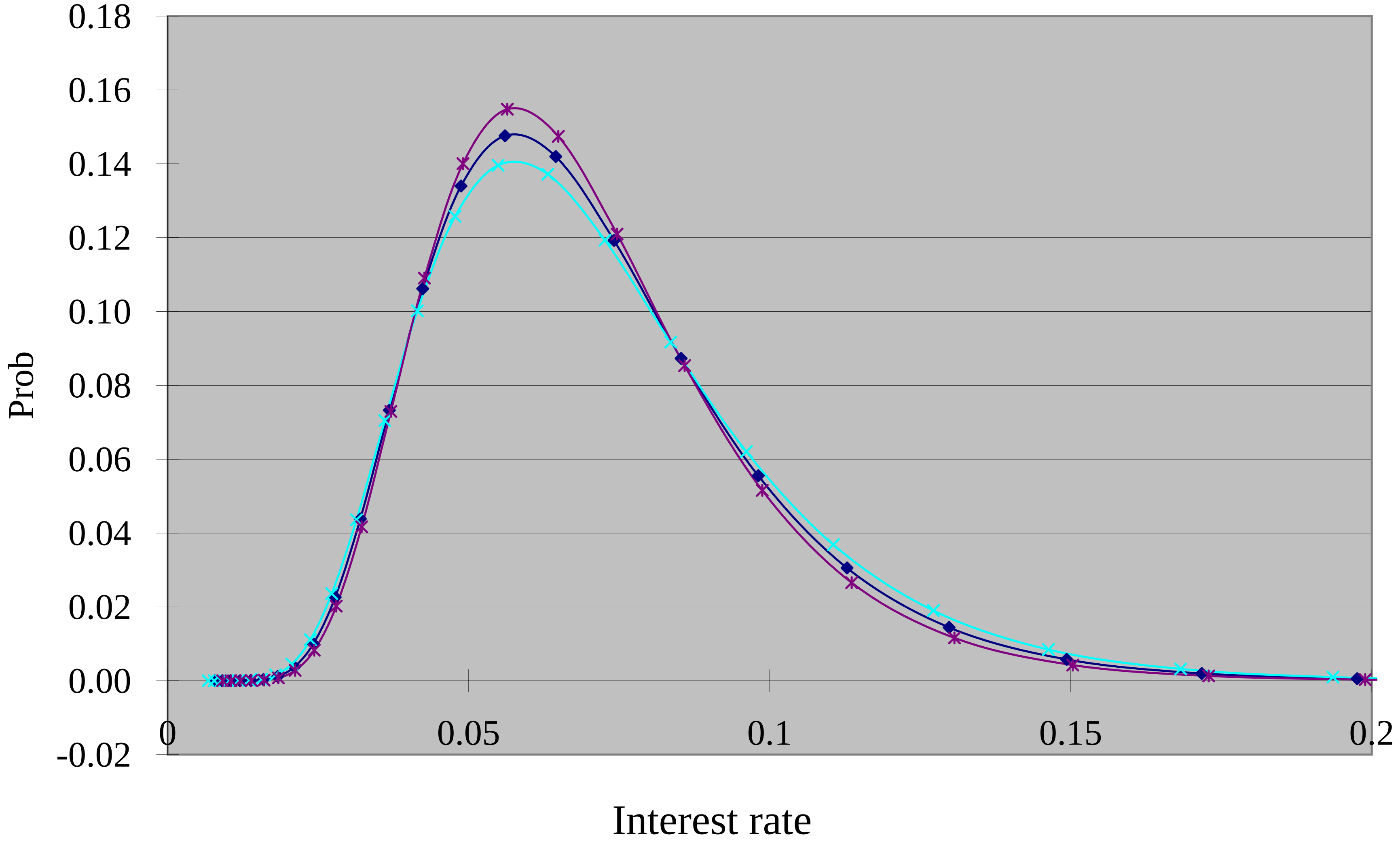

Figure 3
Caplet Prices: Gamma-Adjusted Model

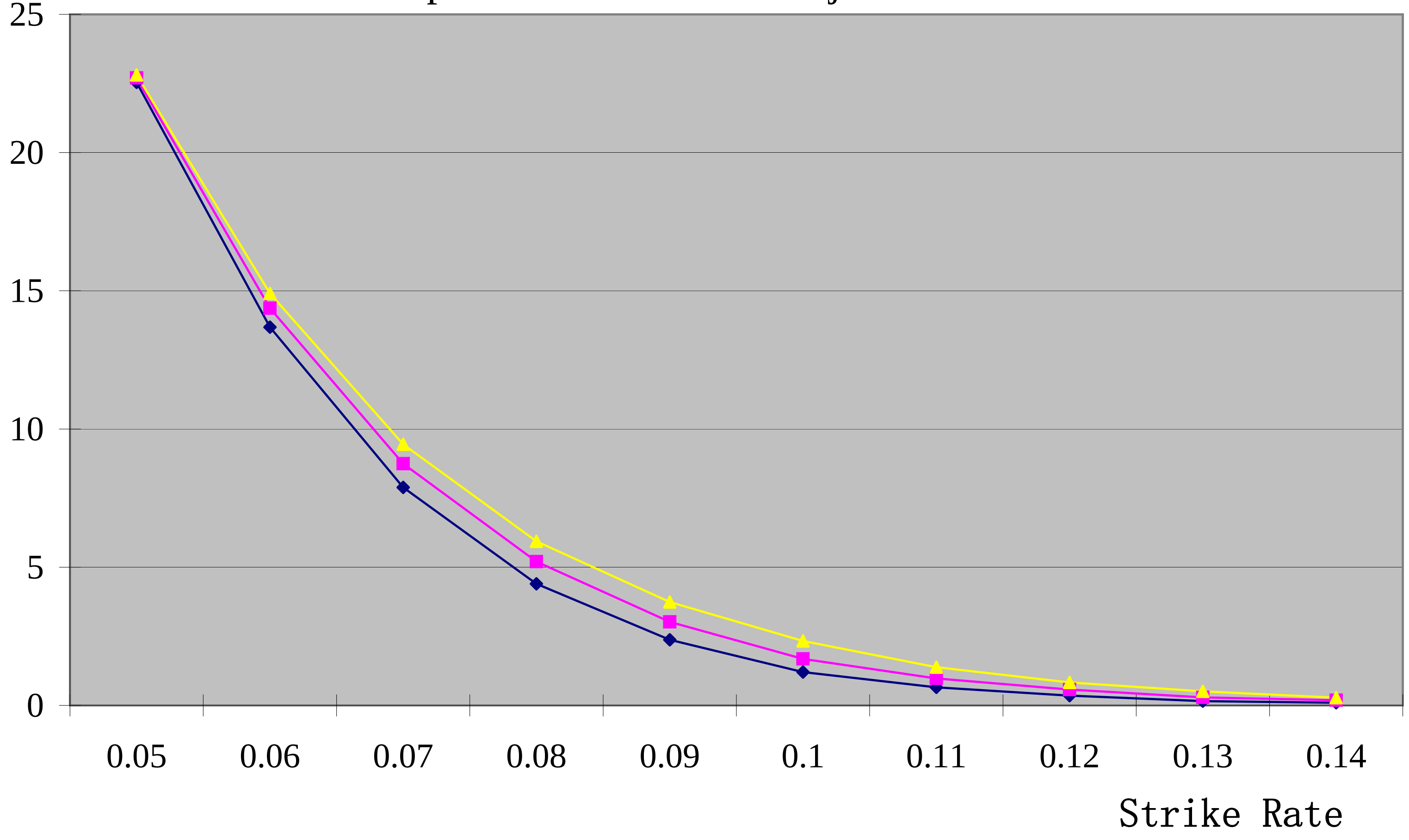

Figure 4: Model Caplet Implied Vols