THE IMPACT OF THE DIFFUSION TERM IN PRICING EUROPEAN OPTIONS ASSUMING STOCHASTIC VOLATILITY

B . L. SHANAHAN

ABSTRACT. While a modelling assumption of stochastic volatility has enjoyed success in recent years, analytic solutions to such specifications can be difficult to obtain. Here we focus on a polynomial which can approximate option prices for volatility specifications which are generalised functions of a mean-reverting process. The approximation is highly accurate under reasonable parameterisation in a European option construct. We then extend the model of Stein and Stein (1991) by altering the diffusion coefficient of the volatility process. That is, we continue to assume an identity process for volatility while removing the assumption that the volatility process is Ornstein-Ühlenbeck by means of its diffusion coefficient. In particular, we examine the use of a power function and find that the power parameter is extremely useful in capturing volatility smile dynamics more realistically[1]. We then show similar effects by assuming an exponential function of volatility in order to provide positivity almost surely.

Contents

1. INTRODUCTION

Recent decades have given rise to a number of sometimes complex and sophisticated methodologies that attempt to resolve empirical issues related to volatility modelling, with a view, in

[1]The Author would like to thank Marti Subrahmanyam for suggesting the line of research taken up in this paper.

particular, to pricing options. Models which assume volatility is stochastic have enjoyed particular success in attempting to explain observed volatility smiles with the Heston (1993) model in particular receiving much attention in empirical studies.

One of the major reasons that stochastic volatility models have become popular is their ability to model stylised features of volatility more succinctly in their bid to better explain volatility smiles. The models of Scott (1987), Stein and Stein (1991), and Heston (1993) in particular have been useful in their ability to model volatility characteristics such as shocks and persistence by assuming a mean reverting process for volatility or a function of volatility. The Heston (1993) model also assumed correlation between the Brownian motions which drive the stock and volatility processes in order to capture another important feature; skewness in the distribution of the asset returns process. That is, zero correlation can only lead to increased kurtosis in the distribution of asset returns and, while this is important in explaining volatility smiles better, empirical evidence supports a skewed distribution for returns, something which cannot be captured properly without correlation.

Given the successes enjoyed by authors including Hull and White (1987), Scott (1987), Stein and Stein (1991), and Heston (1993) in modelling volatility stochastically, therefore, we turn to this framework and look to extend it to more general functional forms of volatility under the consideration that there is trade-off between realism and tractability which is generally realised. Specifically, the functional specifications of volatility in models such as those in Stein and Stein (1991) and Heston (1993) have been predominantly made in order to provide analytic convenience, the cost of which can be realistic option prices. Specifically, we assume the stock price process

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_t$$

where we use the volatility specification $\sigma_t = f(Y_t)$. For the intrinsic volatility process, Y_t, we assume the form

$$dY_t = \kappa(\theta - Y_t)dt + \beta Y_t^{\gamma} dB_t$$

subject to the parameters $\kappa, \theta, \beta \geq 0$ as well as constant correlation coefficient $d\langle B, W\rangle_t = \rho dt$. We are interested in choices for the diffusion power parameter γ where the case of $\gamma = 0$ means that Y_t is an Ornstein-Ühlenbeck (OU) process, while $\gamma = 1/2$ means that Y_t is a Cox Ingersoll Ross (1985) process.

Before elaborating further on the modelling framework and its usefulness, we make the comment that specific examples of the functional form which are popular include $f(Y_t) = \exp(Y_t)$ with $\gamma = \rho = 0$, which is the Scott (1987) model, while $f(Y_t) = |Y_t|$ with $\gamma = \rho = 0$ is the Stein and Stein (1991) model. Similarly, $f(Y_t) = \sqrt{Y_t}$ with $\gamma = 1/2$, $\rho \neq 0$ is the Heston (1993) model. These functional forms, and, in particular, that of Scott (1987) and Stein and Stein (1991) will be investigated in detail, while extensions to more general functional forms will also be outlined. In particular, we are also interested in including correlation for the Scott (1987) and Stein and Stein (1991) models, while revising the functional form to $f(Y_t) = Y_t$ for the Stein and Stein (1991) model.

Extension to multi-dimensional settings also becomes possible, though this is left to future research. While it is true that a multi-dimensional setting for volatility along with careful consideration of the functional form of volatility may be more reliable and accurate for capturing stylised volatility we will focus here predominantly on the possibility of creating greater kurtosis in asset returns by means of the diffusion coefficient of the volatility process. While it is true that consideration of different functional forms for volatility can address this issue somewhat (this follows from the implications of an application of Ito's formula), we consider that a power parameter can offer great explanatory power and, as such, restrict our attention to two functional forms for volatility in order to see this. The first choice of functional form is the identity process, the second is an exponential process, thus ensuring positivity almost surely.

In particular, we find that the choice of the power parameter, γ, along with the functional specification and the sign and size of correlation are all important in determining the shapes of implied volatilities. We achieve this by looking first at the option prices produced, the resultant stock price densities, a summary of distributional moments, and, finally, implied volatilities. The study of the centered moments for the distribution of returns is in the spirit of Das and Sundaram (1999) who, using a variation of the Heston (1993) model, find that it is not always possible to generate enough kurtosis at moderate or long run maturities. Importantly, it will be shown here that the choice for γ can impact upon kurtosis considerably, however, too much explanatory power in γ can lead to large increases in returns variances, thus nullifying a great deal of such kurtosis.

The rest of this paper is as follows: Section 2 provides a theoretical background to the modelling framework presented in Section 3. Section 4 then examines the option prices and densities recovered under different γ assuming the identity process, Y_t. Section 5 does the same for the exponential process, $\exp(Y_t)$. In Section 6 we examine the first four centered distributional moments for each model under different parameters, in the spirit of Das and Sundaram (1999) before examining typical implied volatilities in Section 7.

2. Theoretical Background

Specifically, we will first consider the functional form of volatility to be that in Stein and Stein (1991); the identity process. As has already been noted, this functional form used in conjunction with an Ornstein-Ühlenbeck process for volatility is posed mostly for its ability to provide analytic tractability, though this tractability will degenerate under a changing power diffusion term. So too, has tractability been considered in the assumption of zero correlation made in Stein and Stein (1991), a further restriction that we shall remove. In other words, while the functional form of volatility may be questionable to some degree, the Stein and Stein (1991) model will provide us with a benchmark. The prescribed setting leads us to the specific and revised modelling framework provided by

$$\frac{dS_t}{S_t} = rdt + Y_t dW_t \tag{2.1}$$

for the asset price process, couple with the variance process

$$dY_t = \kappa(\theta - Y_t)dt + \beta Y_t^{\gamma}(\rho dW_t + \sqrt{1-\rho^2} d\tilde{W}_t) \tag{2.2}$$

where the Brownian motions, W_t and $\tilde{W}_t$, are independent.

Once again, this leads us to the consideration of a suitable evaluation methodology given the degeneracy of a Stein and Stein (1991)-type solution due to the diffusion parameter, γ, which, we recall, in their case is set equal to zero. When pricing options within the framework provided by (2.1) and (2.2), one of the most popular approaches is the consideration of the partial differential equation operator (see, for example, Fouque et al 2000) provided by

$$\mathcal{L} \;=\; \frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{1}{2}x^2y^2\frac{\partial^2}{\partial x^2} + \kappa(\theta - y)\frac{\partial}{\partial y} + \frac{1}{2}\beta^2 y^{2\gamma}\frac{\partial^2}{\partial y^2} + \rho\beta x y^{\gamma+1}\frac{\partial^2}{\partial x \partial y} - r \qquad (2.3)$$

We recall that, in order to price options, we wish to solve a general partial differential equation (*pde*) of the form

$$\mathcal{L}u(t,x,y) \;=\; 0 \qquad (2.4)$$

subject to boundary conditions such as $u(T,x,y) = f(x,y)$, $u(0,x,y) = g(x,y)$, and $u(t,B,y) = h(t,y)$. We will consider these in more detail soon, but for the meantime restrict attention to a general method of solution to such partial differential equations.

To that end we recall that, even in the case of zero correlation, analytic solutions to partial differential equations such as those provided by (2.4) are difficult to obtain. In the two dimensional case, that is, where volatility is constant, so that (2.4) reduces to $\mathcal{L}u(t,x) = 0$ subject to $\kappa = \beta = 0$, the partial differential equation is further reduced by the introduction of variables which lead to a two-dimensional heat equation (Elliott and Kopp 2005).

The extra dimension provided by the state variable governing volatility will usually mean that an analytic solution is not attainable (except in specific cases), ultimately requiring some form of numerical approximation. Among those considered along with the approaches of Stein and Stein (1991) and Heston (1993) is that in Fouque et al (2000). We have commented on the method of Stein and Stein (1991) to the extent that they consider only zero correlation and as a result are able to recover the stock price density, though an extension to include correlation is not straightforward. In addition, the approach becomes more complicated under more general volatility specifications arising from the choice of $f(\cdot)$.

As alluded to already, for the method in Heston (1993), the probabilities of the stock price finishing in a specified region along with its expectation restricted to a specified region are found. These probabilities can be considered as estimates of the pricing density and while the form provided must be calculated by use of a numerical procedure, it can be considered closed-form due to its ease of implementation with modern computer software. However, the method used relies heavily on the functional form specification, namely the square root process. The use of the power parameter, γ, will mean that we cannot generally find an analogous solution here, leading to a consideration other methods.

The main idea of the approach of Fouque et al (2000) is to incorporate a flexible functional form by considering an asymptotic expansion of the option price where it follows that the first term in the expansion is the Black Scholes call option price with volatility based on $f(y)$ as in (2.2) and (2.3) with $\gamma = 0$ (that is, OU processes). Here y is instantaneous volatility and the first-order correction term is based on an expansion of the option price in coefficients of the inverse square-root of mean reversion.

The validation of the asymptotic approach is the consideration of high speed mean reversion and that this is, roughly, equivalent to long-horizon maturities. To see this, we note that

$$var(Y_t) \;=\; \frac{\beta^2}{2\kappa}\left(1 - e^{-2\kappa t}\right) \qquad (2.5)$$

which is approximately equal to $\beta^2/(2\kappa)$ for κ large where the link between this approximating value and time is made by the exponential term.

While the methodology is flexible in its ability to handle any suitable functional form of volatility, $f(\cdot)$, there are several issues. To begin with, the Ornstein-Ühlenbeck process is Gaussian and there is only a first-order correction term. This means that the price differences found between options priced in the stochastic volatility setting and those found by standard Black Scholes will be symmetric in correlation about the strike price. Given that the OU process is Gaussian, there is always some probability that it can take on negative values. This can be overcome by the choice of functional form of volatility, thus leading to a skewed distribution, though this will not be captured by the first-order correction term.

In addition, it must be assumed that the speed of mean reversion is numerically high. Even if this is the case, a high enough speed of mean reversion, when volatility of volatility is small, should effectively lead volatility to be approximately constant. This should then result in price differences not too far from a standard Black Scholes option price with effective volatility based on $E[f(Y_t)]$. Given that a high speed of mean reversion is connected to long-run behaviour, it follows that the price differences must be compared to those on a much longer maturity, even in the case of an option which has only two weeks (say) until expiry. Finally, as the correlation between the two Brownian motions tends to zero the price differences disappear completely.

While the methodology will be outlined for general choices of functional form of volatility, $f(Y_t)$, where Y_t is as provided in (2.2) in order to overcome the already mentioned shortcomings. Our specific aim is to determine the amount of contribution made to returns kurtosis by the power parameter, γ, and, therefore, the amount of smile attributable to the modelling framework, all other parameters being equal. The expansion method centres around the volatility of volatility parameter, β, up to second order, essentially meaning that any error is of order β^3. For practical purposes, it is usually found that $\beta \in [0, 1)$ and more usual situations are where β is near 0.1 or 0.2 so that the method is accurate for three to four decimal places at worst. In particular, an increase in the power parameter, γ, will mean that β can be revised down, all else being equal, so that larger values for γ should, theoretically, lead to increased accuracy in the approximation.

Specifically, the polynomial not only includes correlation, but can also be used when correlation is zero. This is achieved by assuming the zero order term, $u_0(\tau, x, y)$, followed by the retrieval of a first order term, $u_1(\tau, x, y)$, which is analogous to than found in Fouque et al (2000). The distinction between the first order term found here and that of Fouque et al (2000) is that the latter is based on expectations with respect to the density of the long run volatility process. We continue the expansion further with a second order term, $u_2(\tau, x, y)$, which is comprised of terms which have zero and non-zero correlation, thus providing flexibility along with increased accuracy.

The resultant polynomial is flexible in its ability to be applied to various settings too. In particular, it will be shown that any function $u(t, x, y)$ which is continuous in its derivatives with respect to t, x, and y satisfying the operator $\mathcal{L}$ provided by (2.3) and, in particular, the adjusted Black Scholes operator

$$\tilde{\mathcal{L}}_{BS} = \frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \kappa(\theta - y)\frac{\partial}{\partial y} + \frac{1}{2}x^2 f(y)^2 \frac{\partial^2}{\partial x^2} - r \tag{2.6}$$

satisfies the conditions of the expansion, where the term taking derivatives in y can be considered as a time-dependent, though deterministic, volatility specification. This approach, therefore, not only includes European options, but also densities as in Stein and Stein (1991) and Heston (1993), thus allowing the retrieval of options based on extrema and infima such as Barrier and Lookback options. While such useful applications will not be pursued here, we will consider the stock price density alone, providing us with much information about the behaviour of such densities under different choices for the power parameter, γ.

3. Methodology - Polynomial Expansions

3.1. **The Heston (1993) Model and Functional Extensions.** While it is the model of Stein and Stein (1991) which is our initial focus, we will use the Heston (1993) model to outline the method of solution due to its relative ease of derivation. To that end, let us recall the specification of the stock and volatility processes as proposed in Heston (1993) which are of the form

$$\frac{dS_t}{S_t} = rdt + \sqrt{Y_t}dW_t$$

and

$$dY_t = \kappa(\theta - Y_t)dt + \beta\sqrt{Y_t}dB_t$$

such that $d\langle B, W\rangle_t/dt = \rho \in [-1, 1]$. We note that the difference between this specification and that in Stein and Stein (1991) is that the latter uses $f(Y_t) = Y_t$, while $\gamma = 0$ instead of $\gamma = 1/2$. We begin by recalling an analogous *pde* operator to that in (2.3) and initially assume, for reasons which will soon become clear, that $\beta = 0$. That is, the operator reduces to the extended Black Scholes operator (2.6). To that end, let us initially consider a European call option, $u(t, x)$, that is, a contingent claim satisfying $\tilde{\mathcal{L}}_{BS}u(t, x) = 0$ subject to $u(T, x, y) = \{x - K\}^+$ for some exercise price, K. The prescribed setting provides the following

Theorem 3.1. *Consider a European contingent claim, $u(\tau, x, y)$, such that $u(0, x, y) = \{x - K\}^+$ and $\tilde{\mathcal{L}}_{BS}u(t, x, y) = 0$. In such a setting, we have*

$$u(\tau, x, y) = x\mathcal{N}(d_1) - Ke^{-r\tau}\mathcal{N}(d_2) \tag{3.7}$$

subject to

$$d_1 = \frac{1}{g(\tau, y)}\left[\ln\frac{x}{K} + r\tau\right] + \frac{1}{2}g(\tau, y) \qquad ; \qquad d_2 = d_1 - g(\tau, y)$$

where

$$g^2(\tau, y) = \theta\tau + \frac{1}{\kappa}(y - \theta)(1 - e^{-\kappa\tau})$$

and where $\tau = T - t$

Proof. Application of the operator, $\tilde{\mathcal{L}}_{BS}$,as provided in (2.6) to (3.7) brings about the result

In other words, the resultant option price is exactly that which follows when one considers volatility to be time-dependent, yet deterministic. For the cases of functional forms other than $f(Y_t) = \sqrt{Y_t}$ we have the following

Corollary 3.2. *Assume the volatility specification $\sigma_t = f(Y_t)$ where*

$$dY_t = \kappa(\theta - Y_t)dt$$

that is, volatility is deterministic. Then

$$g(\tau, y) = \left\{\int_0^\tau f^2(Y_u)du\right\}^{1/2} \tag{3.8}$$

whenever the right hand side is finite with $\tau = T - t$.

Proof. Consider, without loss of generality, $r = 0$, so that

$$\tilde{\mathcal{L}}_{BS}\, u(t,x,y) \;=\; -x\mathcal{N}'(d_1)\frac{\partial}{\partial\tau}g(\tau,y) + \kappa(\theta - y)x\mathcal{N}'(d_1)\frac{\partial}{\partial y}g(\tau,y) + \frac{1}{2g(\tau,y)}xf^2(y)\mathcal{N}'(d_1)$$

Setting the right hand side to zero is equivalent to

$$-2g(\tau,y)\frac{\partial}{\partial\tau}g(\tau,y) + 2g(\tau,y)\kappa(\theta - y)\frac{\partial}{\partial y}g(\tau,y) + f^2(y) \;=\; 0$$

Or, alternatively,

$$-\frac{\partial}{\partial\tau}g^2(\tau,y) + \kappa(\theta - y)\frac{\partial}{\partial y}g^2(\tau,y) + f^2(y) \;=\; 0$$

whence it is clear that $g^2(\tau,y)$ satisfies this *pde*.

Example 3.3. *Consider the functional form of volatility, $f(Y_t) = Y_t$, as in Stein and Stein (1991). Then*

$$g^2(\tau,y) \;=\; \theta^2\tau + \frac{1}{2\kappa}(y-\theta)^2(1 - e^{-2\kappa\tau}) + \frac{2\theta}{\kappa}(y-\theta)(1 - e^{-\kappa\tau}) \tag{3.9}$$

From the preceding result it is clear that we can model deterministic volatility in the option pricing formula (3.7) subject to any reasonable functional specification $f(\cdot)$, whence it is clear that the Black Scholes volatility $g(\tau,y)$ defines a norm in the L^2 sense. In addition, the determinisitic case shows that, under any fixed specification of $f(\cdot)$, any choice of the power parameter, γ, gives rise to the same time-determnistic volatility function, $g(\tau,y)$. This is an artifact of the realisation that the distinction between such processes is captured by the diffusion coefficient, something which will only enter the pricing formula by means of considering the volatility of volatility parameter, β.

Having prescribed our basic pricing structure, we move to include stochastic volatility. To achieve this, we consider an expansion of $u(t,x,y)$ around the volatility of volatility coefficient, β. To this end, assume, without loss of generality, that $r = 0$ as well as

$$u(t,x,y) \;=\; \sum_{n=0}^{\infty}\beta^n u_n(t,x,y) \tag{3.10}$$

and note that, by (2.3),

$$\mathcal{L}u(t,x,y) \;=\; \left(\frac{1}{\beta^2}\mathcal{L}_2 + \frac{1}{\beta}\mathcal{L}_1 + \mathcal{L}_0\right)u(t,x,y) \;=\; 0 \tag{3.11}$$

where notation has be simplified with the use of the operators

$$\mathcal{L}_0 \;=\; \frac{1}{2}y\frac{\partial^2}{\partial y^2}$$

$$\mathcal{L}_1 \;=\; \rho xy\frac{\partial^2}{\partial x\partial y}$$

and

$$\mathcal{L}_2 \;=\; \tilde{\mathcal{L}}_{BS} \;=\; \frac{\partial}{\partial t} + \frac{1}{2}x^2y\frac{\partial^2}{\partial x^2} + \kappa(\theta - y)\frac{\partial}{\partial y}$$

As a technical aside, we comment that the operator, $\mathcal{L}_0$, corresponds to the infinitesimal generator of the CIR process, $\mathcal{L}_1$ is the cross-term operator between the two diffusion processes, picking up on the correlation between the two Brownian motions, and $\mathcal{L}_2$ is the Black Scholes partial differential equation operator under the assumption of zero interest rates and $f(y) = \sqrt{y}$. We see that application of the operator (3.11) to the polynomial expansion provided by (3.10) leads to

$$\mathcal{L}u(t,x,y) = \mathcal{L}_2\beta^{-2}u_0(t,x,y) + \beta^{-1}\left(\mathcal{L}_2u_1(t,x,y) + \mathcal{L}_1u_0(t,x,y)\right) + \sum_{n=0}^{\infty}\beta^n\left(\mathcal{L}_2u_{n+2}(t,x,y) + \mathcal{L}_1u_{n+1}(t,x,y) + \mathcal{L}_0u_n(t,x,y)\right) = 0 \quad (3.12)$$

Setting each coefficient of β equal to zero and begining with the first term, we have

$$\mathcal{L}_2u_0(t,x,y) = 0$$

Given that this equation corresponds to the Black Scholes option pricing operator, it follows that we must have $u_0(t,x,y)$ as provided in Theorem (3.1). For the first-order term, $u_1(t,x,y)$, we have the following

Theorem 3.4. *The first order correction term satisfies*

$$u_1(\tau,x,y) = \rho H_1(\tau,y)\left(x^2u_{xx} + \frac{1}{2}x^3u_{xxx}\right) \quad (3.13)$$

where the expression uses the second and third derivatives of the Black Scholes call option price $u_0(t,x,y)$ with respect to the underlying asset. In addition, the coefficient in τ, y satisfies

$$H_1(\tau,y) = -\frac{1}{\kappa}(y-\theta)\left(\tau e^{-\kappa\tau} - \frac{1}{\kappa}(1-e^{-\kappa\tau})\right) + \frac{\theta}{\kappa}\left(\tau - \frac{1}{\kappa}(1-e^{-\kappa\tau})\right) \quad (3.14)$$

Proof. See appendix

Remark 3.5. For more general functional forms of volatility such as those provided by (3.8) $H_1(\tau,y)$ will satisfy, in the case of the CIR and OU processes, the partial differential equation

$$\tilde{\mathcal{L}}_{BS}H_1(\tau,y) = \frac{\partial}{\partial t}H_1(\tau,y) + \kappa(\theta-y)\frac{\partial}{\partial y}H_1(\tau,y) = -y^{\gamma}f(y)\frac{\partial}{\partial y}g^2(\tau,y) \quad (3.15)$$

We will see the full representation of more general functional forms shortly, but for the meantime we restrict attention to the derivation of the Heston (1993) model.

While it is true that some functional forms of volatility will be easier to implement analytically than others, the general representation is reasonably flexible and useful. The complexity of solution increases as more terms are added to the expansion, though the zero and first order terms will capture most of the explanatory power of the prices obtained in a stochastic volatility setting provided, of course, there is some level of correlation between the two Brownian motions. We now turn to the inclusion of a second-order term as well as possible generalisations of the functional form.

Theorem 3.6. *In the case of the Heston (1993) model, the second-order correction term satisfies*

$$u_2(\tau,x,y) = u_{(2,1)}(\tau,x,y) + \rho^2u_{(2,2)}(\tau,x,y) \quad (3.16)$$

where notation has been simplified with the use of

$$u_{(2,1)}(\tau,x,y) = H_2(\tau,y)(2x^2u_{xx} + 4x^3u_{xxx} + x^4u_{xxxx})$$

and

$$u_{(2,2)}(\tau,x,y) = H_3(\tau,y)\left(4x^2u_{xx} + 23x^3u_{xxx} + 23x^4u_{xxxx} + \frac{13}{2}x^5u_{xxxxx} + \frac{1}{2}x^6u_{xxxxxx}\right) + H_4(\tau,y)\left(x^2u_{xx} + \frac{5}{4}x^3u_{xxx} + \frac{1}{4}x^4u_{xxxx}\right)$$

where

$$H_2(\tau,y) = -\frac{1}{4\kappa^2}(y-\theta)\left(\tau e^{-\kappa\tau} - \frac{1}{2\kappa}(1-e^{-2\kappa\tau})\right) + \frac{\theta}{4\kappa^2}\left(\frac{\tau}{2} - \frac{1}{4\kappa}(3-4e^{-\kappa\tau}+e^{-2\kappa\tau})\right) \quad (3.17)$$

while

$$\begin{aligned} H_3(\tau,y) &= \frac{1}{4\kappa^4}(y-\theta)^2\left(\kappa^2\tau^2e^{-2\kappa\tau} - 2\kappa\tau(e^{-\kappa\tau}-e^{-2\kappa\tau}) + 1 - 2e^{-\kappa\tau} + e^{-2\kappa\tau}\right) \\ &+ \frac{\theta}{2\kappa^4}(y-\theta)\left((2-\kappa^2\tau^2)e^{-\kappa\tau} - (1+\kappa\tau)e^{-2\kappa\tau} - 1 + \kappa\tau\right) \\ &+ \frac{\theta^2}{4\kappa^4}\left(\kappa^2\tau^2 - 2\kappa\tau + 2\kappa\tau e^{-\kappa\tau} + e^{-2\kappa\tau} - 2e^{-\kappa\tau} + 1\right) \end{aligned} \quad (3.18)$$

and

$$H_4(\tau,y) = -\frac{2}{\kappa}(y-\theta)\left(\frac{\tau e^{-\kappa\tau}}{2\kappa}(2+\kappa\tau) - \frac{1}{\kappa^2}(1-e^{-\kappa\tau})\right) + \frac{2\theta}{\kappa^2}\left(\tau(1+e^{-\kappa\tau}) - \frac{2}{\kappa}(1-e^{-\kappa\tau})\right) \quad (3.19)$$

Proof. See appendix

The second-order term highlights one of the most attractive features of the approximation namely, that options can be priced in both the presence of zero and non-zero correlation. The first order term is the larger in magnitude of the two and disappears when correlation is set equal to zero. In the second order term, the component which has the coefficient $H_2(\tau,y)$ is independent of correlation and is the term which accounts for price differences when correlation is set equal to zero.

Moreover, the second-order term comprises, more generally, components which are of order ρ^2 so that, regardless of the sign of correlation observed in practice, this term will always have the same sign. This is in contrast to the first-order term whose sign depends on the sign of correlation. This distinction between the two terms captures the skewness in distribution of the CIR process. The next result provides the conditions required to extend the second-order term to more general functional forms of volatility within the CIR and OU construct

Lemma 3.7. *For more general functional forms of volatility such as those provided by (3.8) $H_2(\tau,y)$ will satisfy, in the case of the CIR and OU processes, the partial differential equation*

$$\frac{\partial}{\partial t}H_2(\tau,y) + \kappa(\theta-y)\frac{\partial}{\partial y}H_2(\tau,y) = -\frac{1}{8}y^{2\gamma}\left(\frac{\partial}{\partial y}g^2(\tau,y)\right)^2 \quad (3.20)$$

In addition, this second term will, more generally have a second component, $H_{2b}(\tau,y)$, such that

$$\frac{\partial}{\partial t}H_{2b}(\tau,y) + \kappa(\theta-y)\frac{\partial}{\partial y}H_{2b}(\tau,y) = -\frac{1}{2}y^{2\gamma}\left[g(\tau,y)\frac{\partial^2}{\partial y^2}g(\tau,y) + \left(\frac{\partial}{\partial y}g(\tau,y)\right)^2\right] \quad (3.21)$$

which can be simplified with the use of

$$g(\tau,y)\frac{\partial^2}{\partial y^2}g(\tau,y) + \left(\frac{\partial}{\partial y}g(\tau,y)\right)^2 = \frac{1}{2}\frac{\partial^2}{\partial y^2}g^2(\tau,y)$$

This results in the inclusion of an extra term,

$$\beta^2 H_{2b}(\tau, y) x^2 u_{xx}$$

Proof. See proof of theorem 3.6

In general, a second function must be included to the component of the second-order term which does not depend on correlation, as the above results shows. However, no other such terms are required in the evaluation of $H_3(\tau, y)$ and $H_4(\tau, y)$. In fact, we can state the following

Theorem 3.8. *Consider general functions such as those involving $g(\tau, y)$, derivatives of this in y, and combinations of these with y^γ, $f(y)$, which we shall denote by $G(\tau, y)$. In addition, consider functions $H(\tau, y)$, such as $H_2(\tau, y)$, $H_{2b}(\tau, y)$, $H_3(\tau, y)$, and $H_4(\tau, y)$ as used in the approximating polynomial. Such functions are required to satisfy*

$$-H_\tau + \kappa(\theta - y)H_y \;=\; -G(\tau, y)$$

where $\tau = T - t$. In such cases, the following equality is satisfied

$$H(\tau, y) \;=\; \int_0^\tau G(u, v(u))du \tag{3.22}$$

where

$$v(u) \;=\; \theta + (y - \theta)e^{\kappa(u-\tau)}$$

In addition, it is clear that such functions tend to 0 as τ tends to zero.

Proof. Note that we have $v(\tau) = y$ as well as the partial derivatives $v_y = \exp(\kappa(u - \tau))$ and $v_\tau = \kappa(y - \theta)\exp(\kappa(u - \tau))$, so that

$$-H_\tau + \kappa(\theta - y)H_y \;=\; -G(\tau, y) + \int_0^\tau \left[v_\tau - \kappa(\theta - y)v_y\right] G_v du \;=\; -G(\tau, y)$$

as required

The great advantage of the aforementioned property is that it be can used to extend to any functional form of volatility without great difficulty. Indeed, even in cases where suitable functions, $G(\tau, y)$, are difficult to integrate analytically in order to satisfy (3.22), modern computer packages such as MATLAB can evaluate these numerically quickly with high precision. Moreover, we have the

Corollary 3.9. *Consider functions $H(\tau, y)$ satisfying (3.22). Then it must be that*

$$\lim_{\kappa\to 0} H(\tau, y) \;=\; \int_0^\tau \lim_{\kappa\to 0} G(u, y)du$$

Proof. It is obvious that $v(u) \to y$ as $\kappa \to 0$ and the argument then follows by continuity of the function G.

Example 3.10. *Consider the term $H_1(\tau, y)$ arising in the first-order term of the Heston (1993) polynomial. The appendix has shown that $G(\tau, y) = y(1 - e^{-\kappa\tau})/\kappa$ so that*

$$G(u, v(u)) \;=\; \frac{1}{\kappa}\left(\theta + (y - \theta)e^{\kappa(u-\tau)}\right)(1 - e^{-\kappa u})$$

and therefore

$$\lim_{\kappa\to 0}\int_0^{\tau} G(u, v(u))du \;=\; y\int_0^{\tau} udu \;=\; \frac{1}{2}y\tau^2$$

as required.

Target Function	$G(\tau, y)$	$G(\tau, y) : \kappa = 0$
$H_1(\tau, y)$	$y^{\gamma} f(y)\frac{\partial}{\partial y}g^2(\tau, y)$	$y^{\gamma} f(y)\frac{\partial}{\partial y}f^2(y)\tau$
$H_2(\tau, y)$	$\frac{1}{8}y^{2\gamma}\left(\frac{\partial}{\partial y}g^2(\tau, y)\right)^2$	$\frac{1}{8}y^{2\gamma}\left(\frac{\partial}{\partial y}f^2(y)\right)^2\tau^2$
$H_{2b}(\tau, y)$	$\frac{1}{4}y^{2\gamma}\frac{\partial^2}{\partial y^2}g^2(\tau, y)$	$\frac{1}{4}y^{2\gamma}\frac{\partial^2}{\partial y^2}f^2(y)\tau$
$H_3(\tau, y)$	$\frac{1}{2}y^{\gamma} f(y)H_1(\tau, y)\frac{\partial}{\partial y}g^2(\tau, y)$	$\frac{1}{2}y^{2\gamma} f^2(y)\left(\frac{\partial}{\partial y}f^2(y)\right)^2\tau^2$
$H_4(\tau, y)$	$2y^{\gamma} f(y)\frac{\partial}{\partial y}H_1(\tau, y)$	$2y^{\gamma} f(y)\frac{\partial}{\partial y}H_1(\tau, y)$

TABLE 1. Summary of functions, $G(\tau, y)$, required to find functional coefficients in τ, y as provided by (3.22) for the generalised CIR and OU processes where $\gamma = 0, 1/2$ respectively

Table 1 provides a summary of the relations between the functions required as coefficients, $H(\tau, y)$, of the approximating polynomial and the functions, $G(\tau, y)$, which they must satisfy given the identity in (3.22). For the Heston (1993) model, where $g^2(\tau, y)$ is provided by theorem (3.1), it is straightforward to show that functions $H_i(\tau, y)$, $i = 1, 2, 3, 4$, as provided in theorems (3.4) and (3.6) satisfy the identity where $\gamma = 1/2$. The third column is the special case of zero mean reversion[2] where a simple integral of such functions with respect to τ will yield the target function.

One of the most inhibitive properties of the first and second order terms is that they include partial derivatives of the corresponding Black Scholes option price, with respect to the underlying asset, up to sixth order. To assist with the tedium of such calculations the following recursion is proposed

[2]An important note is in order with respect to the implementation of the algorithm. Specifically, while the limiting values exist for $\kappa = 0$, programs such as MATLAB will not recognise these limits numerically so that small enough values for the mean reversion parameter, κ, may appear to give rise to prices which grow without bound very quickly. Therefore, care must be taken with implementation and for small enough values of mean reversion ($\kappa < 0.1$, say) the limiting values can be used with reasonable accuracy. Packages, such as MAPLE, for example, will recognise the limits thus allowing prices to be calculated for any mathematically reasonable parameterisation.

Lemma 3.11. *Consider the vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \mathbb{R}^5$. Then the following equality holds for linear combinations and partial derivatives of the Black Scholes option price, $u(t, x, y)$,*

$$\begin{aligned}\alpha_1 x^2 u_{xx} + \alpha_2 x^3 u_{xxx} + \alpha_3 x^4 u_{xxxx} + \alpha_4 x^5 u_{xxxxx} + \alpha_5 x^6 u_{xxxxxx} \\ = \Bigg\{ \alpha_1 + (3\alpha_4 - \alpha_3 - 11\alpha_5)\frac{1}{g^2(\tau, y)} + 3\frac{\alpha_5}{g^4(\tau, y)} \\ -\left(\alpha_2 - \alpha_3 + 2\alpha_4 - 6\alpha_5 + (18\alpha_5 - 3\alpha_4)\frac{1}{g^2(\tau, y)}\right)\left(1 + \frac{d_1}{g(\tau, y)}\right) \\ +\left(\alpha_3 - 3\alpha_4 + 11\alpha_5 - 6\frac{\alpha_5}{g^2(\tau, y)}\right)\left(1 + \frac{d_1}{g(\tau, y)}\right)^2 \\ -(\alpha_4 - 6\alpha_5)\left(1 + \frac{d_1}{g(\tau, y)}\right)^3 + \alpha_5\left(1 + \frac{d_1}{g(\tau, y)}\right)^4 \Bigg\} x^2 u_{xx}\end{aligned} \tag{3.23}$$

Where d_1 is as given in theorem (3.1).

Proof. It is easy to see that

$$u_{xxx} = -\frac{1}{x}u_{xx}\left(1 + \frac{d_1}{g(\tau, y)}\right) \tag{3.24}$$

while

$$u_{xxxx} = \frac{u_{xxx}^2}{u_{xx}} - \frac{1}{x}u_{xxx} - \frac{1}{x^2 g^2(\tau, y)}u_{xx}$$

Similarly,

$$u_{xxxxx} = \frac{u_{xxx}^3}{u_{xx}^2} - \frac{3}{x}\frac{u_{xxx}^2}{u_{xx}} + \frac{2}{x^2}u_{xxx}\left(1 - \frac{3}{2g^2(\tau, y)}\right) + \frac{3}{x^3 g^2(\tau, y)}u_{xx}$$

and

$$\begin{aligned}u_{xxxxxx} = \frac{u_{xxx}^4}{u_{xx}^3} - \frac{6}{x}\frac{u_{xxx}^3}{u_{xx}^2} - \frac{1}{x^2}\frac{u_{xxx}^2}{u_{xx}}\left(\frac{6}{g^2(\tau, y)} - 11\right) + \frac{1}{x^3}u_{xxx}\left(\frac{18}{g^2(\tau, y)} - 6\right) \\ + \frac{1}{x^4}\left(\frac{3}{g^4(\tau, y)} - \frac{11}{g^2(\tau, y)}\right)u_{xx}\end{aligned}$$

Taking linear combinations of these along with the identity

$$\frac{u_{xxx}^n}{u_{xx}^{n-1}} = (-1)^n x^{-n} u_{xx}\left(1 + \frac{d_1}{g(\tau, y)}\right)^n$$

as provided by (3.24) yields the desired result

This recursion may look cumbersome, though it is easily implemented as illustrated in the following

Example 3.12. *Consider the second component in the second-order correction term for the Heston (1993) stochastic volatility model which takes in the linear combination of partial derivatives*

$$2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx}$$

Direct use of (3.23) then says that this is equal to

$$\left\{2 - \frac{1}{g^2(\tau, y)} - 3\left(1 + \frac{d_1}{g(\tau, y)}\right) + \left(1 + \frac{d_1}{g(\tau, y)}\right)^2\right\} x^2 u_{xx}$$

which simplifies to

$$\left\{\frac{d_1^2}{g^2(\tau,y)} - \frac{d_1}{g(\tau,y)} - \frac{1}{g^2(\tau,y)}\right\} \frac{x}{g(\tau,y)}\mathcal{N}'(d_1)$$

3.2. **The Stein and Stein (1991) Model.** We close this section with the polynomial approximation arising from Stein and Stein (1991). The framework established thus far easily extends to this model, though we will simplify the presentation by assuming zero correlation, essentially meaning that we need only find a component of the second order term. It will be easier to evaluate the polynomial arising from consideration of non-zero correlation by the use of numerical procedures coupled with the representations provided in Table 1. Recall, also, that the time-dependent Black Scholes volatility has been provided by (3.9), thus leading to the

Lemma 3.13. *In the model of Stein and Stein (1991), assuming zero correlation, we have the second order polynomial*

$$u_2(t,x,y) = H_2(\tau,y)(2x^2u_{xx} + 4x^3u_{xxx} + x^4u_{xxxx}) + H_{2b}(\tau,y)x^2u_{xx}$$

where

$$\begin{aligned} H_2(\tau,y) = & -\frac{1}{16\kappa^3}(y-\theta)^2(4\kappa\tau e^{-2\kappa\tau} - 1 + e^{-4\kappa\tau}) - \frac{\theta^2}{4\kappa^3}(e^{-2\kappa\tau} - 4e^{-\kappa\tau} - 2\kappa\tau + 3) \\ & - \frac{\theta}{4\kappa^3}(y-\theta)(2\kappa\tau e^{-\kappa\tau} + e^{-3\kappa\tau} - 2 - 2e^{-2\kappa\tau} + 3e^{-\kappa\tau}) \end{aligned}$$

and

$$H_{2b}(\tau,y) = \frac{1}{8\kappa}\left(2\tau - \frac{1}{\kappa}(1 - e^{-2\kappa\tau})\right)$$

Proof. It is clear that the Heston (1993) and Stein and Stein (1991) polynomial approximations differ only in their functional coefficients, $H_i(\tau,y)$, $i = 1,2,3,4$. Given the deterministic Black Scholes volatility, $g(\tau,y)$, as provided by (3.9) combined with theorem 3.8 and Table 1 where $\gamma = 0$, the result follows.

Corollary 3.14. *The coefficients of the second order term in τ, y for the model of Stein and Stein (1991) have the following limits*

$$\lim_{\kappa\to 0} H_2(\tau,y) = \frac{1}{6}y^2\tau^3$$

and

$$\lim_{\kappa\to 0} H_3(\tau,y) = \frac{1}{4}\tau^2$$

Proof. Successive applications of L'Hopital's rule provides the result

The preceding result is included not only for the purpose of completion, but also to help when one wishes to implement the algorithm in a computer package which does not recognise these limits numerically for small values of κ when required.

4. The Impact of a Changing Diffusion Power Parameter, γ, as an Extension to the Stein and Stein (1991) Model

Having outlined the evaluation methodology for two of the more popular stochastic volatility models, we now use Table 1 to look at the implications of changing the parameter γ in the volatility specification

$$dY_t = \kappa(\theta - Y_t)dt + \beta Y_t^{\gamma} dB_t$$

where correlation between B_t and the Brownian motion driving the stock price process, W_t, is equal to $\rho \in [-1, 1]$ along with the functional form of volatility $f(Y_t) = Y_t$. Given it is the impact of γ that we are most interested in, we consider all other parameters constant as summarised in Table 2, and it is easier to use numerical packages to evaluate $H_i(\tau, y)$; $i = 1, 2, 2b, 3, 4$. Realistically, parameters including that for mean reversion, κ, and that for volvol, β, should possibly be revised to compensate for changes in γ, though we will find it most useful to hold these fixed in order to isolate the impact of changing γ.

Identity Process Parameters	Y_t
Initial value, y	0.1
Long-run mean, θ	0.1
Mean reversion, κ	3.0
Volatility of volatility, β	0.1
Correlation, ρ	-0.5

Table 2. **Input parameters for the identity process stochastic volatility model:** Other inputs are $r = 0.05$, $\tau = 0.5$, $K = \$100$.

It is noted that instantaneous volatility, y, has been assumed to equal its long-run mean, θ, and, subsequently, the constant Black Scholes reference volatility, $\bar{\sigma} = 0.1$. If we instead assumed that volatility began either above or below its long-run mean then this would create greater kurtosis in one of the tails of the log returns distribution, potentially producing an implied volatility smirk. Once again, we ignore this possibility and focus solely on the excess kurtosis created in both tails by consideration of γ.

Having set the scene, we now turn to Figure 1 to examine the impact of changes in γ where it takes on values in the set $[-0.75, 0.75]$, where we recall that we have the Stein and Stein (1991) model for $\gamma = 0$. Specifically, it is the impact of the first and second order option pricing terms, $u_1(\tau, x, y)$ and $u_2(\tau, x, y)$, which are illustrated. That is, the improvement in the price of the options over the zero order term, the Black Scholes option price, $u_0(\tau, x, y)$, are being illustrated when the strike price, K, is fixed at \$100 along with interest rate, $r = 0.05$, and option maturity, $\tau = 0.5$, relative to changes in the initial asset price, $x \in [80, 120]$.

The first thing to notice is the magnitudes of the improvement in price. For the values $\gamma \in [-0.25, 0.75]$ all stochastic volatility option prices offer improvements over Black Scholes restricted to the range $[-\$1, \$1]$. However, as γ becomes more negative (taking on values in the set $[-0.75, -0.5]$), these changes in price increase in absolute value exponentially, with the largest

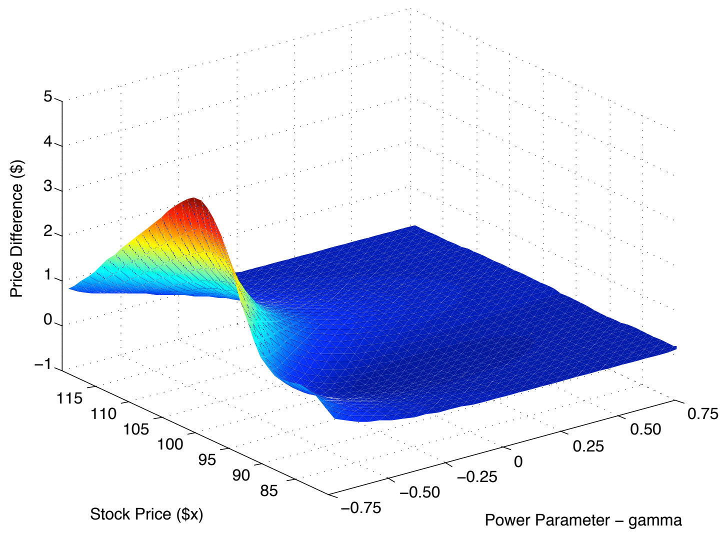

FIGURE 1. **Differences between stochastic volatility and Black Scholes European call option prices:** The functional form of volatility is $\sigma_t = Y_t$, while parameters are $y = \theta = 0.1, \kappa = 3, \beta = 0.1, \rho = -0.5, r = 0.05, \tau = 0.5$ years, $K = \$100$. Black Scholes volatility is $\bar{\sigma} = 0.1$

price improvement over Black Scholes being close to $4.

When we have the values $\gamma \in [-0.5, 0.75]$, the pattern in price changes is reasonably constant: At the money option prices for stochastic volatility are more expensive than for Black Scholes. Then, as options move further in and out of the money, the price differences become negative before rising again and converging to Black Scholes.

Figure 2 illustrates the impact of γ on the changes in option prices compared to Black Scholes by consideration of the first derivative. That is,

$$\frac{\partial}{\partial \gamma} u(\tau, x, y) = \frac{\partial}{\partial \gamma} u_1(\tau, x, y) + \frac{\partial}{\partial \gamma} u_2(\tau, x, y)$$

What becomes clear is that the option price becomes much more sensitive to γ as $\gamma \downarrow -1$. For $\gamma \in [-0.5, 0.75]$, the option prices are much less sensitive for $\gamma > 0$.

In particular, Figure 3 indicates that in the limit, $\gamma \downarrow 0^+$, at the money option prices are more sensitive while in the limit, $\gamma \uparrow 0.75^-$, out of the money options are more sensitive than in the money options. However, the impact of $\gamma \to 0$ in general appears to have the greatest positive implications for volatility smiles.

Next, we examine the stock price densities as recovered by the use of (see Breeden and Litzenberger 1979) the second order derivative

$$\frac{\partial^2}{\partial^2 K} u(\tau, x, y; \kappa, \theta, \beta, \rho, r, K) = f_{S_T}(z = K; x, y, \tau)$$

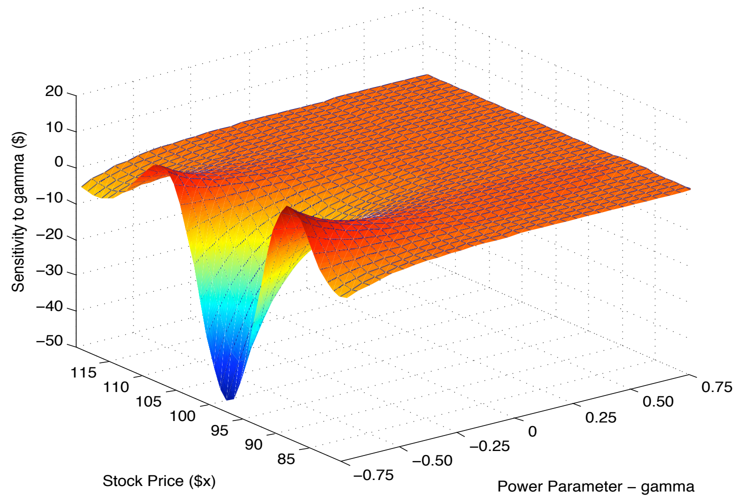

Figure 2. **Sensitivity of stochastic volatility European call option prices to power parameter, γ:** The functional form of volatility is $\sigma_t = Y_t$, while parameters are $y = \theta = 0.1, \kappa = 3, \beta = 0.1, \rho = -0.5, r = 0.05, \tau = 0.5$ years, $K = \$100$. Black Scholes volatility is $\bar{\sigma} = 0.1$

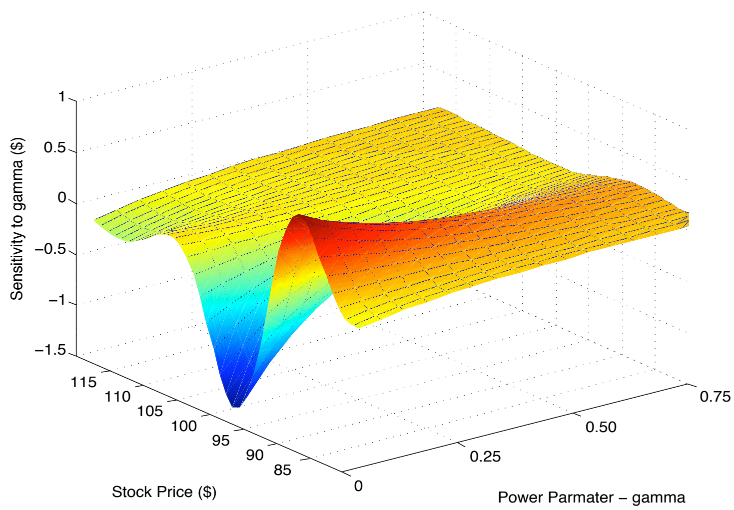

Figure 3. **Sensitivity of stochastic volatility European call option prices to power parameter, γ, a closer inspection:** The functional form of volatility is $\sigma_t = Y_t$, while parameters are $y = \theta = 0.1, \kappa = 3, \beta = 0.1, \rho = -0.5, r = 0.05, \tau = 0.5$ years, $K = \$100$. Black Scholes volatility is $\bar{\sigma} = 0.1$

that is, the density of the terminal stock price, S_T, evaluated at the strike price K given initial values τ, x, y and the parameter space $\{\kappa, \theta, \beta, r\}$.

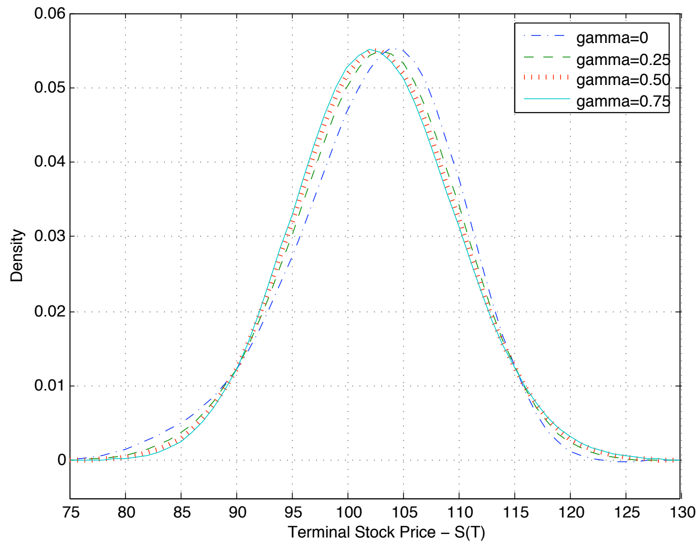

FIGURE 4. **Terminal stock price density assuming stochastic volatility:** The functional form of volatility is $\sigma_t = Y_t$, while parameters are $y = \theta = 0.1$, $\kappa = 3$, $\beta = 0.1$, $\rho = -0.5$, $r = 0.05$, $\tau = 0.5$ years, $x = \$100$.

Figure 4 shows the resultant density for power parameters in the set $\gamma \in \{0, 0.25, 0.5, 075\}$, from which a number of items become apparent. Firstly, as $\gamma \downarrow 0^+$, the density is becoming negative in parts which is attributable to the non-zero probability that volatility is negative. That is, the introduction of correlation to the Stein and Stein (1991) model has led to an unrealistic density. In addition, negative powers for γ exacerbate this problem meaning we must restrict attention to $\gamma > 0$. The intuition for such results is clear: For a fixed time horizon, τ, the volatility paths are almost surely less than one. This means that increasing γ in the diffusion coefficient for the volatility process, Y_t, is reducing its impact, thus leading to volatility which becomes deterministic as γ becomes large. On the other hand, forcing the power parameter, γ, to be smaller increases the contribution of the diffusion coefficient and, subsequently, the probability that volatility can become negative. In other words

$$\lim_{\gamma \downarrow -\infty} \mathbb{P}[\, Y_t < 0 \,\text{for some}\, t \,] \;=\; 1$$

The second item which becomes apparent is the way in which γ impacts on the skewness of the distribution of returns. In particular, with consideration that we have fixed $\rho = -0.5$, the distribution is most negatively skewed for $\gamma = 0$, the Stein and Stein (1991) model. However, as $\gamma \uparrow 1^-$, the distribution becomes more symmetric, thus negating the skewness produced by the correlation coefficient, ρ. This, again, is an artefact of volatility becoming deterministic as γ increases.

To examine closer the impact of the diffusion power parameter, γ, on the density for log returns, we look to its first derivative in Figure 5. That is,

$$\frac{\partial^3}{\partial\gamma\partial^2 K} u(\tau, x, y; \kappa, \theta, \beta, \rho, r, K)$$

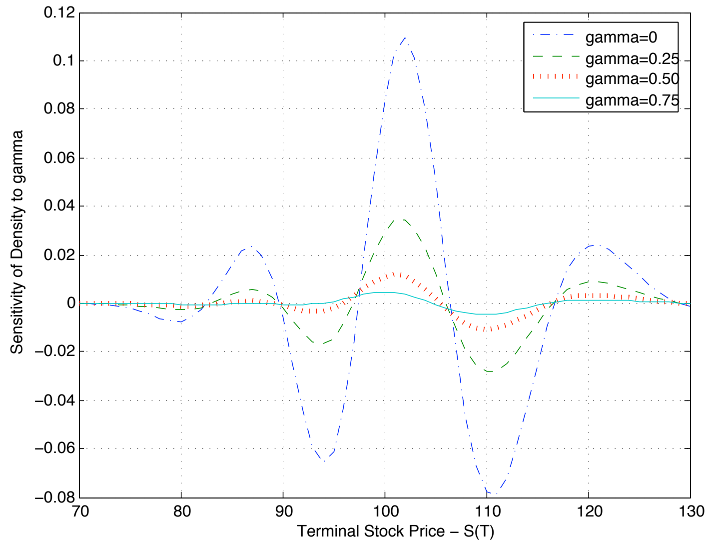

Figure 5. **Sensitivity of terminal stock price density to power parameter, γ, assuming stochastic volatility:** The functional form of volatility is $\sigma_t = Y_t$, while parameters are $y = \theta = 0.1$, $\kappa = 3$, $\beta = 0.1$, $\rho = -0.5$, $r = 0.05$, $\tau = 0.5$ years, $x = \$100$.

It is clear that the densities become more sensitive to γ when this value is closer to zero. Essentially, for the second derivative,

$$\frac{\partial^4}{\partial\gamma^2\partial^2 K}u(\tau, x, y; \kappa, \theta, \beta, \rho, r, K)$$

the distribution is placing more mass at its mean, less in its immediate tails and then more again as the tails become thinner and thinner, the diminishing skewness for larger values of γ becoming more apparent and these effects are occurring at a diminishing rate.

5. The Impact of a Changing Diffusion Power Parameter, γ, as an Extension to the Scott (1987) Model

Having investigated the implications of changing γ for $\sigma_t = Y_t$, we now look the implications of changing the parameter γ in the volatility specification

$$dY_t = \kappa(\theta - Y_t)dt + \beta Y_t^\gamma dB_t$$

where correlation between B_t and the Brownian motion driving the stock price process, W_t, is equal to $\rho \in [-1, 1]$ along with the functional form of volatility $f(Y_t) = \exp(Y_t)$, as an extension of the Scott (1987) model. The reason for examining this second specification is the problems caused by the sample paths for the identity process, $f(Y_t) = Y_t$. We recall that, in that case, in the limit, $\gamma \downarrow 0^+$, volatility can become negative with increasing probability, thus leading to unrealistic densities for the returns process. The idea with this second specification is to enforce volatility to be positive, almost surely, though some considerations are needed in order to make useful comparison of both functional specifications.

Firstly, the value for y and, subsequently those for θ, κ, β, must be revised so that instantaneous volatility is equal to 0.1 as before. In other words, we now require $\exp(y) = 0.1$ meaning that the paths of the process, Y_t, must be mostly negative. Because this is the case, the diffusion power parameter, γ, must be restricted to integer values otherwise Y_t will become a complex-valued process.

In addition, $Y_0 = y = -2.306$ and, because the absolute value for this is greater than one, it is clear that integer values for γ in the set $\{\dots, -3, -2, -1\}$ will lead to less variability in the sample paths for Y_t meaning that it won't be able to generate enough kurtosis in the log returns process. For these reasons, we restrict attention to positive integer values for γ with the understanding that larger values will lead to more variability in the sample paths of Y_t and, subsequently, greater kurtosis in the log returns process.

Exponential Process Parameters	$\exp(Y_t)$
Initial value, y	-2.3026
Long-run mean, θ	-2.3026
Mean reversion, κ	5.0
Volatility of volatility, β	0.1
Correlation, ρ	-0.5

TABLE 3. **Input parameters for the exponential process stochastic volatility model:** Other inputs are $r = 0.05$, $\tau = 0.5$, $K = \$100$.

Parameters for the functional form, $f(Y_t) = \exp(Y_t)$, are summarised in Table 3 and we specifically examine the choices $\gamma \in \{0, 1, 2, 3, 4\}$. The case of $\gamma = 0$ can be seen as an extension to the model of Scott (1987) by the introduction of correlation. Other choices for γ, therefore, can be seen as an extension to this model by change of diffusion coefficient. In particular, Figure 6 indicates the price differences between stochastic volatility and Black Scholes option prices. As expected, increasing γ leads to more variability in the sample paths of volatility and, therefore, greater price differences.

Beginning with the model of Scott (1987), we find that all stochastic volatility option prices are within \$0.02 of Black Scholes, relatively symmetric about the strike price $K = \$100$, and positive for in the money options, and negative for out of the money options due to the assumption of negative correlation. As γ increases there are two noticeable effects: First, the price differences are spreading to include more in and out of the money options. Second, the difference in pricing reverses as γ increases. Specifically, each integer increase in γ is reversing the effect of correlation. For $\gamma = 1$ it is mostly out of the money options which are more expensive than Black Scholes, in the money options are mostly cheaper. For $\gamma = 2$ this trend reverses to that for $\gamma = 0$, that is, out the money options are mostly cheaper, in the money options are more expensive than Black Scholes. This follows because the diffusion term is having a positive impact on sample paths by virtue of the even powers, so we expect that odd powers are of more interest, thus keeping the sample paths negative with greater probability. Also, as γ increases, the magnitudes of positive price differences increases relative to negative price differences to the point that all prices are more expensive than Black Scholes for $\gamma = 4$.

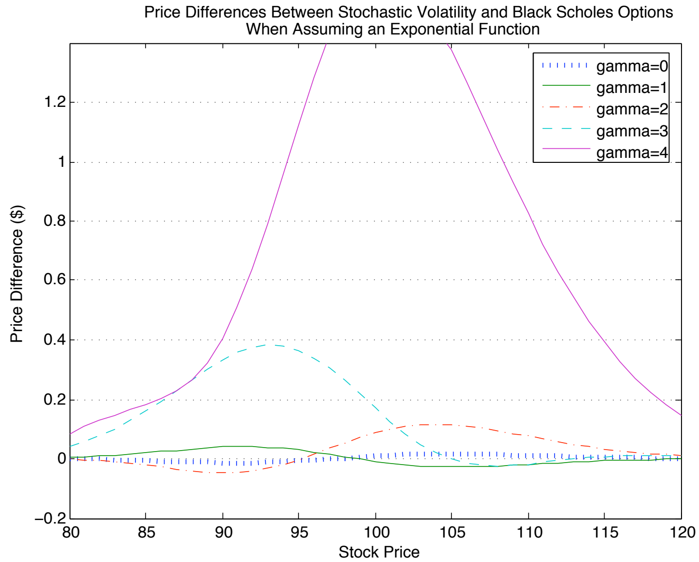

Figure 6. **Differences between stochastic volatility and Black Scholes European call option prices:** The functional form of volatility is $\sigma_t = \exp(Y_t)$, while parameters are $y = \theta = -2.3026$, $\kappa = 5$, $\beta = 0.1$, $\rho = -0.5$, $r = 0.05$, $\tau = 0.5$ years, $K = \$100$. Black Scholes volatility is $\bar{\sigma} = 0.1$

Figure 7 reiterates these points where we see the respective return densities. In particular, the density for the Scott (1987) model is the least skewed with densities for $\gamma = 1, 3$ being positively skewed while densities for $\gamma = 2, 4$ are negatively skewed, the amount of skewness increasing in absolute value as γ increases. For odd values of γ we find that the densities are positively skewed due to the almost sure negativity of the sample paths of Y_t, so that it is necessary to reverse the sign of correlation in order to create negative skewness. Another effect to notice is that one of the tails becomes fatter as γ increases, the other becoming thinner due to correlation. The density corresponding to $\gamma = 4$ is the most unrealistic, though in all of these cases (as opposed to those for $f(Y_t) = Y_t$), all have the properties of densities and it appears that larger values for γ will correspond to greater curvature in implied volatility smiles.

6. Moment Summaries

Our initial investigation has considered the use of a power parameter in the diffusion coefficient of intrinsic volatility, Y_t, when the functional form of volatility is either $f(Y_t) = Y_t$ or $f(Y_t) = \exp(Y_t)$, and it appears that the price differences and stock price densities recovered suggest that it is the second specification with increasing powers for γ which will provide the most potential in providing implied volatility smiles. This follows, we recall, from the fact that the use of an identity process supports lower powers for γ, though for $\gamma \leq 0$ unrealistic densities will be recovered. In other words, for $\gamma > 0$, there may not be enough curvature captured in implied volatilities.

To investigate the potential further, we now examine the variance, skewness, and kurtosis of the

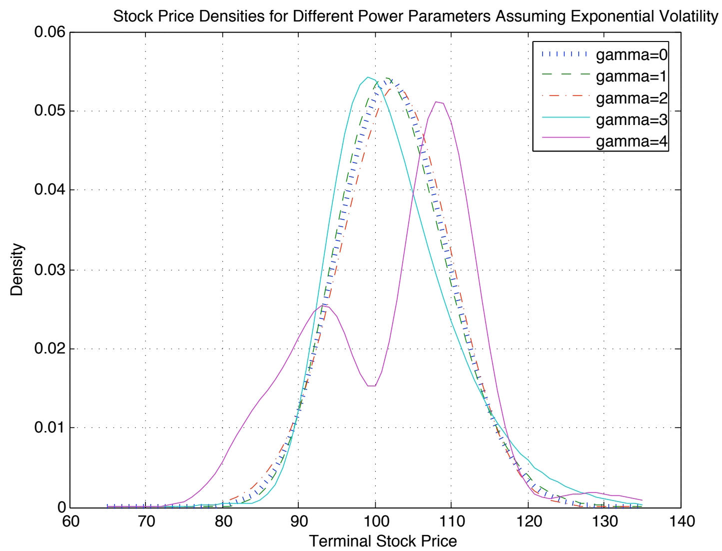

FIGURE 7. **Terminal stock price density assuming stochastic volatility:** The functional form of volatility is $\sigma_t = \exp(Y_t)$, while parameters are $y = \theta = -2.3026$, $\kappa = 5$, $\beta = 0.1$, $\rho = -0.5$, $r = 0.05$, $\tau = 0.5$ years, $x = \$100$.

terminal stock price distribution for each functional specification, as in the spirit of Das and Sundaram (1999). The objective, therefore, of this section is to examine the potential of each specification, under different parameters, to provide enough excess kurtosis at different maturities in order to explain volatility smiles more accurately. In order to retrieve each of the four centered moments required, we again turn to a second order derivative of the option price with respect to the strike price, K. Recall that this is the value of the stock price density evaluated at the strike (that is, $S_T = K$) so that integrating with respect to K yields the desired results. Numerical integration packages using MATLAB are employed in each case.

To broaden the analysis conducted thus far, we consider two different maturities, three different values for correlation, and two values for volatility of volatility. Specifically, we consider the terminal stock price in two weeks and again at six months. Each of these is evaluated at $\rho \in \{-0.5, 0, 0.5\}$ in order to better see the impact of correlation on skewness. The two volatility of volatility choices made are $\beta = 0.1$ and $\beta = 0.3$, allowing us to get more of an indication of the kurtosis created for each of the power parameters.

6.1. **The Identity Process.** The results obtained for variance, skewness, and kurtosis in each of the twelve cases can be seen in Tables 4 - 9, the average stock prices are \$100.1925 and \$102.5315 for the time horizons $T = 1/26$ and $T = 0.5$ respectively. We see that in almost all cases that kurtosis increases as γ decreases. The three exceptions are when $\gamma = 0$, $T = 0.5$, $\beta = 0.3$ for each of $\rho = -0.5, 0, 0.5$.

We recall that at $\gamma = 0$ there is a potential for the density to become negative, however, for γ close to zero the greatest kurtosis will attain. This problem, however, is not arising over the shorter

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	3.9164	-0.2147	3.1149	57.0683	-0.3237	3.5263
$\gamma = 0.25$	3.8755	-0.0984	3.0280	53.6620	-0.1374	3.1261
$\gamma = 0.50$	3.8641	-0.0303	3.0048	52.7665	0.0040	3.0108
$\gamma = 0.75$	3.8614	0.0086	3.0008	52.5858	0.0925	3.0176

TABLE 4. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 2 with $x = \$100$, $r = 0.05$, $\rho = -0.5$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	4.4392	-0.6011	4.0251	102.2397	-0.2836	4.6003
$\gamma = 0.25$	4.0338	-0.3775	3.3891	67.1246	-0.4174	4.4354
$\gamma = 0.50$	3.9102	-0.2011	3.1174	56.5582	-0.2979	3.5737
$\gamma = 0.75$	3.8737	-0.0903	3.0292	53.5219	-0.1190	3.1442

TABLE 5. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 2 except $\beta = 0.3$ with $x = \$100$, $r = 0.05$, $\rho = -0.5$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	3.9306	0.0620	3.1426	58.7439	0.2678	3.7102
$\gamma = 0.25$	3.8835	0.0599	3.0506	54.6080	0.2324	3.3192
$\gamma = 0.50$	3.8686	0.0592	3.0203	53.3001	0.2192	3.1606
$\gamma = 0.75$	3.8639	0.0589	3.0106	52.8865	0.2148	3.1064

TABLE 6. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 2 with $x = \$100$, $r = 0.05$, $\rho = 0$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	4.4819	0.0821	3.8991	107.1341	0.3915	4.0773
$\gamma = 0.25$	4.0579	0.0675	3.3657	69.9103	0.3300	4.2247
$\gamma = 0.50$	3.9237	0.0617	3.1295	58.1391	0.2632	3.6621
$\gamma = 0.75$	3.8813	0.0598	3.0462	54.4167	0.2305	3.2972

TABLE 7. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 2 except $\beta = 0.3$ with $x = \$100$, $r = 0.05$, $\rho = 0$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

term horizon of two weeks, that is, the impact of the larger volvol is negated by time. In addition, increases in volvol, β, lead to larger increases in skewness over time than do smaller values of volvol. Interestingly, kurtosis also increases as distributions become more positively skewed though values of correlation in the range $[-1, 0]$ are generally preferred.

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	3.9450	0.3375	3.2405	60.4639	0.8474	4.3412
$\gamma = 0.25$	3.8916	0.2178	3.1016	55.5715	0.5999	3.7137
$\gamma = 0.50$	3.8731	0.1486	3.0468	53.8403	0.4346	3.3802
$\gamma = 0.75$	3.8664	0.1093	3.0246	53.1896	0.3376	3.2252

TABLE 8. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 2 with $x = \$100$, $r = 0.05$, $\rho = 0.5$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	4.5250	0.7587	4.2629	112.4266	1.1079	4.8455
$\gamma = 0.25$	4.0820	0.5105	3.5783	72.8531	1.0896	5.1596
$\gamma = 0.50$	3.9373	0.3242	3.2371	59.7796	0.8301	4.3634
$\gamma = 0.75$	3.8890	0.2099	3.0991	55.3334	0.5840	3.7060

TABLE 9. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 2 except $\beta = 0.3$ with $x = \$100$, $r = 0.05$, $\rho = 0.5$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

Regarding skewness, the general trend is that distributions become more skewed as γ decreases. The two exceptions to this are at $\gamma = 0$, $T = 0.5$, $\beta = 0.3$ for each of $\rho = -0.5, 0.5$, again for the reason that, over this time horizon, the density becomes unrealistic.

In particular, all summary statistics for skewness and kurtosis confirm that densities become closer to normal as γ increases. In other words, volatility is becoming deterministic over time for larger power parameters because this increase leads to a dampening of the diffusion coefficient. The most resounding example of this is that where $\beta = 0.1$, $T = 1/26$, $\rho = -0.5$.

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	3.8571	0.0312	3.0020	52.6307	0.1589	3.0434
$\gamma = 1$	3.8673	0.1227	3.0328	53.4334	0.3398	3.2290
$\gamma = 2$	3.8867	-0.0849	3.0293	54.3696	-0.0443	3.0543
$\gamma = 3$	4.0698	0.3777	3.3266	66.9311	0.7282	3.9389
$\gamma = 4$	4.8628	-0.4664	3.6411	116.0001	-0.1518	2.9150

TABLE 10. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 3 with $x = \$100$, $r = 0.05$, $\rho = -0.5$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

6.2. **The Exponential Process.** The centered moments obtained using the exponential process are summarised in Tables 10 - 15. As before the means are \$100.1925 and \$102.5315 for the time horizons $T = 1/26$ and $T = 0.5$ respectively. Regarding kurtosis, one item becomes immediately apparent: The amounts generated are not as large as those produced using the identity process. This finding would appear to suggest that this second model is not as capable as the first in providing volatility smiles. However, there is one major factor which is resulting in less kurtosis and

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	3.8648	-0.0235	3.0080	53.0449	0.0581	3.0269
$\gamma = 1$	3.9294	0.2476	3.1510	57.6194	0.5742	3.6740
$\gamma = 2$	4.1671	-0.3216	3.3503	72.1467	-0.2227	3.5274
$\gamma = 3$	5.6690	0.6634	4.0025	171.4372	0.6377	2.6586
$\gamma = 4$	13.1430	-0.2758	2.5409	645.3886	0.0768	0.7541

Table 11. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 3 except $\beta = 0.3$ with $x = \$100$, $r = 0.05$, $\rho = -0.5$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	3.8585	0.0588	3.0074	52.7641	0.2132	3.0855
$\gamma = 1$	3.8641	0.0590	3.0133	53.1240	0.2153	3.1063
$\gamma = 2$	3.8940	0.0598	3.0431	55.0725	0.2255	3.2027
$\gamma = 3$	4.0527	0.0638	3.1814	63.7615	0.2722	3.6184
$\gamma = 4$	4.9018	0.0778	3.5298	119.7343	0.2938	2.8754

Table 12. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 3 with $x = \$100$, $r = 0.05$, $\rho = 0$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	3.8690	0.0591	3.0181	53.4430	0.2170	3.1234
$\gamma = 1$	3.9198	0.0605	3.0677	56.6823	0.2332	3.2702
$\gamma = 2$	4.1892	0.0668	3.2766	74.2189	0.2796	3.4553
$\gamma = 3$	5.6176	0.0840	3.5640	161.9283	0.2895	2.4410
$\gamma = 4$	13.2592	0.0835	2.3066	656.1752	0.1674	0.7028

Table 13. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 3 except $\beta = 0.3$ with $x = \$100$, $r = 0.05$, $\rho = 0$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

this is the increased variances. In particular, when $\gamma = 4$, the amount of kurtosis present it at its smallest relative to all other choices for γ. The exceptions to this are the cases, (i) where $\beta = 0.1$, $\rho = 0.5$, for both time horizons, and (ii), $\beta = 0.1$, $\rho = -0.5$, $T = 1/26$. The reason for this effect appears too be the huge increases in variance, that is, the distributional tails are spreading further, though at a decreasing rate.

Larger volvol is also leading to decreased kurtosis for the cases where $\gamma = 3$, a further indication that the coupling of larger γ with larger β will not be sufficient. In other words, for the exponential model, the power parameter must be chosen carefully and larger values of it should possibly be offset by a smaller volvol parameter.

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	3.8599	0.0865	3.0150	52.8981	0.2678	3.1361
$\gamma = 1$	3.8609	-0.0046	3.0029	52.8167	0.0916	3.0198
$\gamma = 2$	3.9014	0.2047	3.0966	55.7876	0.4965	3.4894
$\gamma = 3$	4.0358	-0.2502	3.1890	65.3235	-0.1959	3.1759
$\gamma = 4$	4.9412	0.6200	3.8242	123.6072	0.7390	3.2102

Table 14. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 3 with $x = \$100$, $r = 0.05$, $\rho = 0.5$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

	variance	skewness	kurtosis	variance	skewness	kurtosis
$\gamma = 0$	3.8731	0.1420	3.0468	53.8472	0.3785	3.2947
$\gamma = 1$	3.9102	-0.1260	3.0652	55.7720	-0.0997	3.1544
$\gamma = 2$	4.2115	0.4568	3.5120	76.4005	0.8048	4.0867
$\gamma = 3$	5.5668	-0.4899	3.8451	166.4781	-0.0349	2.3763
$\gamma = 4$	13.3782	0.4556	2.5742	668.2099	0.2822	0.7695

Table 15. **Centered Moments for the Stock Price Distribution for the Exponential Process:** Parameters are as in Table 3 except $\beta = 0.3$ with $x = \$100$, $r = 0.05$, $\rho = 0.5$, $\tau = 2$ weeks (left panel), $\tau = 0.5$ (right panel)

7. Implied Volatilities

Having begun our initial investigation to the impacts on option prices by use of the power parameter, γ under two functional specifications for stochastic volatility we now look to the implied volatilities which are produced. For the identity process, $f(Y_t) = Y_t$, we consider powers in the set $\gamma \in \{0, 0.25, 0.50, 0.75\}$ when $S_0 = x = \$100$, $r = 0.05$, $\tau = 0.5$ years, all other parameters provided in Table 2, and strike prices in the range $K \in [85, 115]$. In the second model, the use of the exponential process, $f(Y_t) = \exp(Y_t)$, we consider powers in the set $\gamma \in \{0, 1, 2, 3, 4\}$ when $S_0 = x = \$100$, $r = 0.05$, $\tau = 0.5$ years, all other parameters provided in Table 3, and strike prices in the range $K \in [85, 115]$.

Figure 7 shows the implied volatilities for the identity process model with $\gamma = 0$ corresponding to the Stein and Stein (1991) model. In all four cases, the shapes of the curves are more likely to be described by smirks or skews than smiles. However, it is clear that the amount of curvature increases as the power parameter, γ, decreases. We reiterate that this effect is somewhat different in nature to the choice of the volvol parameter, β. While it is true that increasing β will increase variability in the sample paths for Y_t, the difference that the parameter γ produces is the increases path-dependence in the process Y_t. In particular, for $\gamma = 0$, Y_t is a Gaussian process and therefore symmetric in distribution. However, as γ, increases this property no longer holds and the process Y_t will have a skewed distribution.

Figure 9 illustrates the implied volatilities for the exponential process, $f(Y_t) = \exp(Y_t)$, with $\gamma = 0$ corresponding to the Scott (1987) model. As expected, the even and odd powers are creating a range of shapes, that corresponding to $\gamma = 4$ being the most unrealistic. The choice of $\gamma = 3$ has

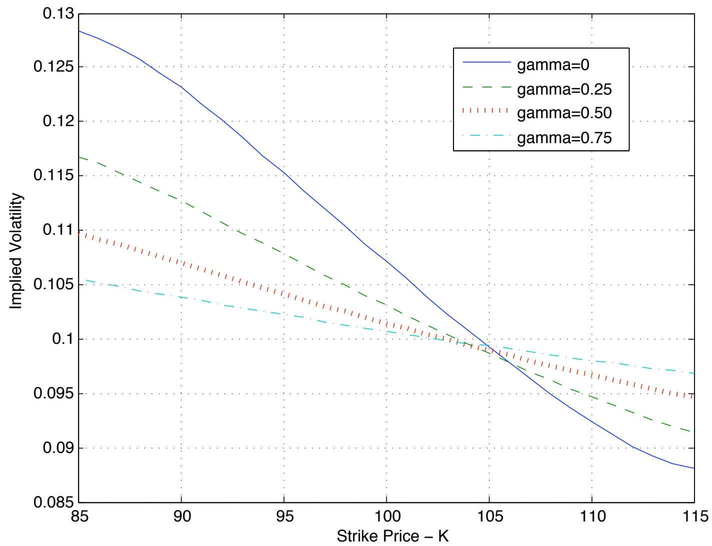

Figure 8. **Implied Volatilities:** The functional form of volatility is $\sigma_t = Y_t$, while parameters are $y = \theta = 0.1$, $\kappa = 5$, $\beta = 0.1$, $\rho = -0.5$, $r = 0.05$, $\tau = 0.5$ years, $x = \$100$.

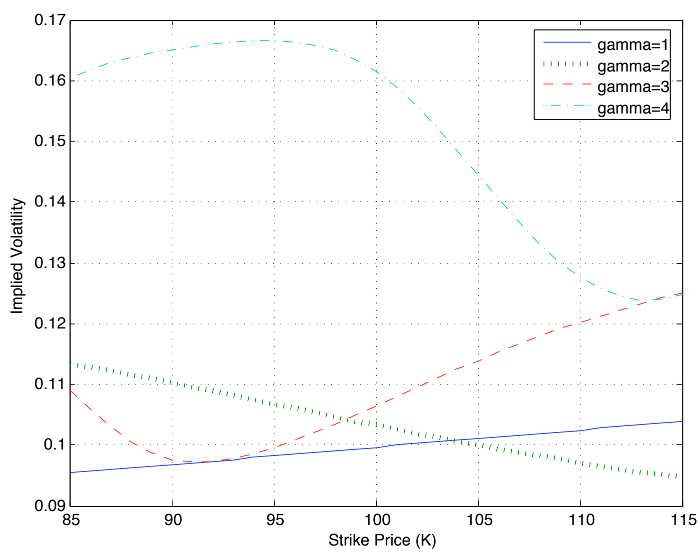

Figure 9. **Implied Volatilities:** The functional form of volatility is $\sigma_t = \exp(Y_t)$, while parameters are $y = \theta = -2.3026$, $\kappa = 5$, $\beta = 0.1$, $\rho = -0.5$, $r = 0.05$, $\tau = 0.5$ years, $x = \$100$.

created a smile and it should be recalled that the extent to which this is true depends on a number of factors. These include the speed of mean reversion, κ, the long-run mean of the volatility process, θ, the instantaneous volatility, y, and its relation to long run mean, and the sign and size of correlation. In other words, choices such as the distance of instantaneous volatility to its long run mean will change the nature of the smile. Similarly, lowering the speed of mean reversion will create more variability, thus increasing the amount of curvature present.

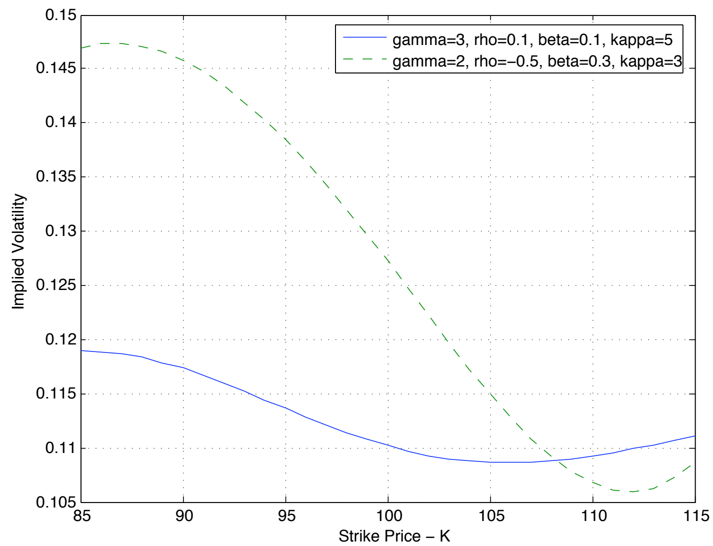

FIGURE 10. **Implied Volatility:** The functional form of volatility is $\sigma_t = \exp(Y_t)$, while parameters are $y = \theta = -2.3026$, $r = 0.05$, $\tau = 0.5$ years, $x = \$100$.

Given the success of the choice $\gamma = 3$ in providing a smile, we now look to this potential in the case where the log returns distribution is negatively skewed. To accomplish this, we set $\rho = 0.1$, recalling that the paths of Y_t are negative more often than not almost surely, and that, due to the magnitudes of the sample paths, this will impact on the skewness in the distribution further. The implied volatility smile recovered from this parameterisation is seen in Figure 10. We note that the shape is almost the reverse of that seen when correlation was set to $\rho = -0.5$.

Figure 10 also provides a volatility smile for an alternate parameterisation when $\gamma = 2$ and $\rho = -0.5$. The reason for this choice stems from Table 11 where it is recalled that, for $\kappa = 5$, $\beta = 0.3$, there is negative skew and excess kurtosis. In the example here we lower mean reversion to $\kappa = 3$ to create more variability in the process Y_t and it can be seen that the smile has its turning point deeper out of the money than that produced for $\gamma = 3$ and also with pronounced curvature. Furthermore, Tables 10 - 15 indicate that the general shapes should be preserved over different maturity horizons.

8. CONCLUSION

This paper has confirmed that key considerations in the modelling of stochastic volatility such as correlation and functional choice for volatility are important in explaining implied volatility smiles. While the functional form has been investigated in only little detail (and deserves extension in its own right), the choice of a diffusion power parameter has also been shown to be important.

In particular, the impact of the term βY_t^γ as a diffusion coefficient in the intrinsic volatility process, Y_t, has been modelled under the choices of functional form of volatility $f(Y_t) = Y_t$ and $f(Y_t) = \exp(Y_t)$, the importance of the choice for γ being related to path-dependence in the volatility process and the subsequent impact of this on the distribution for Y_t.

In the first model, $f(Y_t) = Y_t$, it was shown that decreasing values for γ increase the curvature in smiles and smirks owing to the size of the sample paths of the process, Y_t. In particular, over most reasonable time horizons, the paths of volatility are less than one almost surely so that increasing the power parameter, γ, reduces the impact of the diffusion coefficient, and that larger values for γ correspond to the process Y_t becoming closer to being deterministic. That is, values for γ closer to zero are more able to explain curvature, although care must be taken to ensure that the volatility process does not become negative in its sample paths with reasonable probability.

In the case of the second model, $f(Y_t) = \exp(Y_t)$, we had to restrict γ to taking on integer values to ensure that the process Y_t was real-valued. This follows because instantaneous volatility, y, and the long-run mean of volatility, θ, must be chosen to be negative so that $\exp(y)$ provides a realistic measure of initial volatility. In addition, the absolute value of these parameters will almost always be greater than one so that increasing γ increases the variability in the sample paths.

There are two sets for γ which deserve separate attention; even integers and odd integers. For the case of even integers, the power parameter has the effect of enforcing the diffusion coefficient to be positive, that is, Y_t can oscillate about the origin over time and this effect will be exacerbated in increasing γ to the extent that the volatility process has a distribution with much excess kurtosis, thus impacting on the tails of the density for the log returns process unproportionally.

For odd powers of γ, the process Y_t may still oscillate though to a much lesser extent than for even powers, thus resulting in more negative sample paths. This means that negative correlation between the Brownian motions leads to a positive skew for the distribution for returns, the reverse occurs for positive correlation. It was seen that larger and odd powers for γ produce the most curvature in implied volatilities, the choice of $\gamma = 3$ leading to a smile under the parameters assumed here. Yet again, the extent of this smile will depend largely on inputs such as y, θ, and correlation, ρ.

Given the success of the choice of parameter $\gamma = 3$ in explaining a volatility smile, we used this parameterisation again, however, changed the correlation parameter, ρ, in order to recover a negatively skewed distribution for asset returns. Once again a smile shape was produced, almost the reverse in shape to that produced for negative correlation, and it follows that the power parameter, γ has a lot of explanatory power in producing implied volatility smiles within a stochastic volatility modelling specification.

In general, care should be taken with the choice for γ in either model. We have not only seen that certain ranges for it lead to volatility to becoming more deterministic, but also that allowing it to have too much explanatory power can have too great an influence on option prices produced. In particular, if the power parameter has too much influence then it is possible that the variance of the stock price distribution will grow unproportionally resulting in an important loss of kurtosis. This, in turn, results in a flattening of the implied volatilities for options further in and out of the money. However, it has been seen that certain choices for γ lead to greater curvature in implied volatilities.

It appears that the sign and size of correlation, ρ will have a large influence on the position of the turning point of volatility smiles. All else being equal (including the functional form) positively skewed distributions have the most hope of producing an implied volatility smile, though we have seen examples where this has occurred for negative skewness which is important because negative skewness is supported empirically.

The functional form of volatility obviously has important implications for the amount of kurtosis and skewness in returns distributions and should be carefully considered. An area of future research might be the investigation of a functional choice, $f(\cdot)$, such that, $\sigma_t < f(Y_t) < \exp(Y_t)$ for all t almost surely. Specifically, the process, Y_t may not be able to generate enough kurtosis, while $\exp(Y_t)$ may produce to much variance.

In sum, however, the choices for functional form, correlation, and the power parameter all have important implications for implied volatility smiles. One key modelling consideration which this paper has not addressed is the correct specification of the market price of volatility risk which should arise not only out of consideration of the diffusion term, βY_t^γ, but also in terms of the functional specification. In other words, all parameters have been assumed to model such risk indirectly and an area of future research would be to solidify a meaningful method of incorporating a realistic expression for volatility risk.

References

[1] Breeden, D. T., Litzenberger, R. H. (1978) Price of state-contingent claims implicit in option prices. *Journal of Business* 51(4), 621-651

[2] Cox, J.C., Ingersoll, J.E., Ross, S.A. (1985) A theory of the term structure of interest rates. *Econometrica* 53, 385-407

[3] Das, S. R., Sundaram, R. K. (1999) Of Smiles and Smirks: A Term Structure Perspective. *Journal of Financial and Quantitative Analysis* 34 (2), 211-239

[4] Elliott, R. J., Kopp, P.E. (2005) *Mathematics of Financial Markets*. Springer Finance

[5] Fouque, J.-P., Papanicolaou, G.C., Sircar, K.R. (2000) *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press, Cambridge

[6] Heston, S.L. (1993) A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financial Studies* 6, 327-343

[7] Hull, J., White, A. (1987) The Pricing of Options on Assets with Stochastic Volatilities. *Journal of Finance* 42, 281-300

[8] Scott, L. (1987) Option Pricing When the Variance Changes Randomly: Theory, Estimation, and an Application. *Journal of Financial and Quantitative Analysis* 22(4), 419-438

[9] Stein, E.M., Stein, J.C. (1991) Stock price distributions with stochastic volatility: an analytic approach. *Rev. Financial Studies* 4, 727-468

9. Appendix

9.1. **Proof of Theorem 3.4.** The expansion in (3.12) provides the coefficient of β,

$$\tilde{\mathcal{L}}_{BS} u_1(t,x,y) + \mathcal{L}_1 u_0(t,x,y) \;=\; 0$$

However,

$$\tilde{\mathcal{L}}_1 u_0(\tau,x,y) \;=\; \rho xy \frac{\partial}{\partial y}\mathcal{N}(d_1) \;=\; -\rho xy \frac{1}{g(\tau,y)} d_2 \mathcal{N}'(d_1) \frac{\partial}{\partial y} g(\tau,y) \;=\; -\rho y d_2 x^2 u_{xx} \frac{\partial}{\partial y} g(\tau,y)$$

Therefore, we look for α_1, α_2, and $h(g(\tau,y))$ such that

$$\alpha_1 x^2 u_{xx} + \alpha_2 x^3 u_{xxx} \;=\; -h(g(\tau,y)) d_2 x^2 u_{xx}$$

whence it becomes clear that we must have $\alpha_1 = 1$, $\alpha_2 = 1/2$, $h(g(\tau, y)) = 1/2g(\tau, y)$. That is,

$$\tilde{\mathcal{L}}_{BS}\, u_1(\tau, x, y) \;=\; -2\rho y g(\tau, y)\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right)\frac{\partial}{\partial y} g(\tau, y) \;=\; -\rho y\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right)\frac{\partial}{\partial y} g^2(\tau, y)$$

Alternatively,

$$\tilde{\mathcal{L}}_{BS}\, x^n \frac{\partial^n}{\partial x^n} u_0(\tau, x, y) \;=\; 0$$

And, because $\tilde{\mathcal{L}}_{BS}$ is linear in its derivatives in t and y, this means there exists some function, $H_1(\tau, y)$, such that

$$\tilde{\mathcal{L}}_{BS}\, u_1(\tau, x, y) \;=\; \rho\left(x^2 u_x x + \frac{1}{2}x^3 u_{xxx}\right)\tilde{\mathcal{L}}_{BS}\, H_1(\tau, y)$$

where

$$\tilde{\mathcal{L}}_{BS}\, H_1(\tau, y) \;=\; -y\frac{\partial}{\partial y} g^2(\tau, y)$$

The function $H_1(\tau, y)$ provided in (3.14) satisfies this equality.

9.2. **Proof of Theorem 3.6.** The expansion in (3.12) provides the coefficient of β^2,

$$\tilde{\mathcal{L}}_{BS}\, u_2(\tau, x, y) \;=\; \tilde{\mathcal{L}}_{BS}\, [u_{2,1}(\tau, x, y) + u_{2,2}(\tau, x, y)] \;=\; -\mathcal{L}_0 u_0(\tau, x, y) - \mathcal{L}_1 u_1(\tau, x, y)$$

where we will let $u_{2,1}(\tau, x, y)$ correspond to the operator, $\mathcal{L}_0$, and $u_{2,2}(\tau, x, y)$ correspond to $\mathcal{L}_1$ We look first to the right part of the right hand side of this where we find that

$$-\mathcal{L}_0 u_0(\tau, x, y) \;=\; -\frac{1}{2}y\frac{\partial}{\partial y}\left[x\mathcal{N}'(d_1)g_y\right] \;=\; -\frac{1}{2}y\left(g g_{yy} + d_1 d_2 (g_y)^2\right)x^2 u_{xx}$$

where g_y and g_{yy} denote the usual first and second derivatives of $g(\tau, y)$ with respect to y respectively. Because of the term involving $d_1 d_2$, we look for the constants, $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 x^2 u_{xx} + \alpha_2 x^3 u_{xxx} + \alpha_3 x^4 u_{xxxx} \;=\; h(g(\tau, y), d_1 d_2)x^2 u_{xx}$$

for some suitable function, h. It is not difficult to show that the choice of $\alpha_1 = 2$, $\alpha_2 = 4$, $\alpha_4 = 1$, leads to

$$\mathcal{L}_0 u_0(\tau, x, y) \;=\; \frac{1}{2}y\left(g g_{yy} + (g_y)^2\right)x^2 u_{xx} + \frac{1}{2}y(g_y g)^2(2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx})$$

However, for the Heston (1993) model, $g g_{yy} + (g_y)^2 = 0$, leaving us with

$$-\mathcal{L}_0 u_0(\tau, x, y) \;=\; -\frac{1}{2}y(g_y g)^2(2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx})$$

That is,

$$\tilde{\mathcal{L}}_{BS}\, u_{2,1}(\tau, x, y) \;=\; -\frac{1}{2}y(g_y g)^2(2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx}) \;=\; (2x^2 u_{xx} + 4x^3 u_{xxx} + x^4 u_{xxxx})\tilde{\mathcal{L}}_{BS}\, H_2(\tau, y)$$

and we find $H_2(\tau, y)$ provided in (3.17) satisfies this equality.
In order to recover $u_{2,2}(\tau, x, y)$, we now look to the term, $-\mathcal{L}_1 u_1(\tau, x, y)$. To that end, we have

$$\rho\frac{\partial}{\partial y}H_1(\tau, y)\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right) \;=\; \rho\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right)\frac{\partial}{\partial y}H_1(\tau, y) + \rho H_1(\tau, y)\frac{\partial}{\partial y}\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right) \tag{9.25}$$

The first term on the right hand side is straightforward, for we must then have

$$\rho^2 xy\frac{\partial}{\partial x}\left(x^2 u_{xx} + \frac{1}{2}x^3 u_{xxx}\right)\frac{\partial}{\partial y}H_1(\tau, y) \;=\; 2\rho^2 y\left(x^2 u_{xx} + \frac{5}{4}x^3 u_{xxx} + \frac{1}{4}x^4 u_{xxxx}\right)\frac{\partial}{\partial y}H_1(\tau, y)$$

That is, there exists the function, $u_{2,2,1}(\tau, x, y)$, such that

$$\tilde{\mathcal{L}}_{BS}\, u_{2,2,1}(\tau, x, y) = 2\rho^2 \left(x^2 u_{xx} + \frac{5}{4} x^3 u_{xxx} + \frac{1}{4} x^4 u_{xxxx} \right) \tilde{\mathcal{L}}_{BS}\, H_4(\tau, y)$$

from which we find that (3.19) satisfies this equality. We now turn to the second term on the right side of (9.25). In light of theorem 3.4, we seek

$$\rho H_1(\tau, y) \frac{\partial}{\partial y} \frac{1}{2g(\tau, y)} d_2 x^2 u_{xx}$$

and find this equal to

$$\rho H_1(\tau, y) \frac{1}{2g^2} g_y \left(d_1 d_2^2 - d_1 - 2d_2 \right) x^2 u_{xx}$$

Because the expression involves $d_1 d_2^2$ we look for the constants, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and find that

$$\frac{1}{x^2 u_{xx}} g^3(\tau, y)(4x^2 u_{xx} + 14x^3 u_{xxx} + 8x^2 u_{xxxx} + x^5 u_{xxxxx}) \;=\; d_1 d_2^2 - d_1 - 2d_2$$

That is, the choice of $\alpha_1 = 4$, $\alpha_2 = 14$, $\alpha_3 = 8$, and $\alpha_4 = 1$, leads to

$$\rho H_1(\tau, y) \frac{\partial}{\partial y} \left(x^2 u_{xx} + \frac{1}{2} x^3 u_{xxx} \right) \;=\; \frac{1}{2} \rho H_1(\tau, y) g g_y (4x^2 u_{xx} + 14x^3 u_{xxx} + 8x^2 u_{xxxx} + x^5 u_{xxxxx})$$

Because we are using the operator, $\mathcal{L}_1$, we need to differentiate with respect to x and multiply by $\rho x y$. Doing so says that

$$\begin{aligned} \tilde{\mathcal{L}}_{BS}\, u_{2,2,2}(\tau, x, y) &= \rho^2 y H_1(\tau, y) g g_y \left(4x^2 u_{xx} + 23x^3 u_{xxx} + 23x^2 u_{xxxx} + \frac{13}{2} x^5 u_{xxxxx} + \frac{1}{2} x^6 u_{xxxxxx} \right) \\ &= \rho^2 \tilde{\mathcal{L}}_{BS}\, H_3(\tau, y) \left(4x^2 u_{xx} + 23x^3 u_{xxx} + 23x^2 u_{xxxx} + \frac{13}{2} x^5 u_{xxxxx} + \frac{1}{2} x^6 u_{xxxxxx} \right) \end{aligned}$$

from which we find that (3.18) satisfies this equality.

STRIKE (K)	$\gamma = 0$	$\gamma = 0.25$	$\gamma = 0.50$	$\gamma = 0.75$	Black Scholes
85	17.1573	17.1288	17.1168	17.1117	17.1066
86	16.2040	16.1666	16.1510	16.1439	16.1369
87	15.2567	15.2094	15.1892	15.1799	15.1703
88	14.3164	14.2583	14.2329	14.2210	14.2084
89	13.3845	13.3150	13.2839	13.2691	13.2531
90	12.4626	12.3814	12.2444	12.3265	12.3068
91	11.5226	11.4599	11.4170	11.3959	11.3724
92	10.6569	10.5532	10.5047	10.4807	10.4535
93	9.7780	9.6646	9.6112	9.5846	9.5542
94	8.9190	8.7975	8.7403	8.7117	8.6788
95	8.0831	7.9559	7.8962	7.8664	7.8322
96	7.2738	7.1437	7.0834	7.0534	7.0192
97	6.4950	6.3652	6.3062	6.2773	6.2445
98	5.7504	5.6246	5.5690	5.5423	5.5126
99	5.0439	4.9259	4.8758	4.8526	4.8275
100	4.3792	4.2727	4.2302	4.2114	4.1923
101	3.7600	3.6683	3.6349	3.6215	3.6093
102	3.1894	3.1152	3.0920	3.0844	3.0798
103	2.6699	2.6148	2.6024	2.6009	2.6042
104	2.2034	2.1680	2.1663	2.1708	2.1817
105	1.7906	1.7743	1.7827	1.7926	1.8105
106	1.4314	1.4324	1.4496	1.4644	1.4882
107	1.1244	1.1399	1.1644	1.1829	1.2115
108	0.8670	0.8937	0.9235	0.9448	0.9767
109	0.6556	0.6899	0.7230	0.7459	0.7798
110	0.4859	0.5241	0.5585	0.5820	0.6166
111	0.3530	0.3916	0.4256	0.4487	0.4828
112	0.2515	0.2877	0.3199	0.3419	0.3744
113	0.1762	0.2079	0.2371	0.2573	0.2875
114	0.1221	0.1477	0.1733	0.1913	0.2187
115	0.0844	0.1034	0.1249	0.1405	0.1648

TABLE 16. **Stochastic Volatility Option Prices:** $f(Y_t) = Y_t$, parameters are provided in Table 2 along with $x = \$100$, $r = 0.05$, $\tau = 0.5$

9.3. **Option Price Tables.**

STRIKE (K)	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$	Black Scholes
85	17.1038	17.1223	17.1158	17.3102	17.1066
86	16.1325	16.1585	16.1451	16.4053	16.1369
87	15.1641	15.1992	15.1760	15.5151	15.1703
88	14.1998	14.2460	14.2099	14.6412	14.2084
89	13.2416	13.3005	13.2488	13.7849	13.2531
90	12.2921	12.3650	12.2958	12.9480	12.3068
91	11.3544	11.4420	11.3551	12.1323	11.3724
92	10.4324	10.5343	10.4317	11.3398	10.4535
93	9.5303	9.6454	9.5315	10.5718	9.5542
94	8.6530	8.7789	8.6609	9.8291	8.6788
95	7.8055	7.9388	7.8261	9.1116	7.8322
96	6.9930	7.1293	7.0332	8.4180	7.0192
97	6.2204	6.3545	6.2871	7.7461	6.2445
98	5.4923	5.6186	5.5917	7.0936	5.5126
99	4.8123	4.9256	4.9495	6.4579	4.8275
100	4.1836	4.2788	4.3616	5.8377	4.1923
101	3.6080	3.6812	3.8275	5.2329	3.6093
102	3.0866	3.1348	3.3457	4.6453	3.0798
103	2.6190	2.6411	2.9138	4.0788	2.6042
104	2.2041	2.2001	2.5288	3.5386	2.1817
105	1.8399	1.8113	2.1873	3.0309	1.8105
106	1.5234	1.4731	1.8856	2.5623	1.4882
107	1.2512	1.1830	1.6201	2.1382	1.2115
108	1.0195	0.9377	1.3875	1.7630	0.9767
109	0.8242	0.7333	1.1845	1.4386	0.7798
110	0.6612	0.5656	1.0079	1.1651	0.6166
111	0.5264	0.4303	0.8549	0.9401	0.4828
112	0.4160	0.3227	0.7227	0.7594	0.3744
113	0.3264	0.2387	0.6091	0.6177	0.2875
114	0.2543	0.1741	0.5115	0.5086	0.2187
115	0.1967	0.1253	0.4282	0.4257	0.1648

Table 17. **Stochastic Volatility Option Prices:** $f(Y_t) = \exp(Y_t)$, parameters are provided in Table 3 along with $x = \$100$, $r = 0.05$, $\tau = 0.5$