Extended Nonparametric American Option Pricing.

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Abstract

A nonparametric method of pricing American options was recently developed that requires only historical underlying price data (Alcock and Carmichael, 2008). We derive an extension to this method to include conditioning information from a small number of observed option prices. This additional information improves the overall accuracy of the method and enables pricing of illiquid options in an incomplete market. We explore the statistical properties of both the original method and our extension using a series of simulation studies. The original method slightly outperforms Black-Scholes estimators and numerical estimators (Crank-Nicholson) using historical volatility. In contrast, the extended method presented here produces significant reductions in mean pricing errors. These reductions are most dramatic for out-of-the-money options; a result that is consistent with empirical results for related entropic European option price methodologies.

Key words: Nonparametric option pricing, American options, S&P100 Index, OEX options.

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1 Introduction and Background

American style options are the most common equity and index options traded on the Chicago Board of Options Exchange (Stentoft, 2004). American options are also among the most challenging to price, due to their early exercise feature. Alcock and Carmichael (2008) (AC08) recently developed a nonparametric method of pricing American style options in a complete market that requires only historical returns for the underlying security. The AC08 method is the nexus of Stutzer's (1996) nonparametric Canonical method for pricing European style options and the least-squares algorithm of Longstaff and Schwartz (2001). In this article we present an extension to the original AC08 method that incorporates conditioning information from observed option prices. Incorporating an option price constraint ensures that all price sensitive factors are incorporated into option price estimates. We test this extension in a series of controlled simulation experiments. These simulation experiments demonstrate the accuracy and precision of the extended pricing method, as well as determining the optimal implementation.

Ait-Sahalia and Lo (1998) motivate the use of nonparametric methods for pricing options on several levels. First, nonparametric methods are by their very nature adaptive and capable of responding to structural shifts in the data-generating process in ways that parametric models are not. Second, nonparametric methods provide greater flexibility for use in estimating the prices of a wide range of derivative securities (encompassing a broad scope of asset-pricing dynamics). Finally, nonparametric methods are generally simple and computationally efficient to implement. Herein lies the attraction of nonparametric approaches that may explain their growing prominence in the option pricing literature.

A nonparametric method that only uses historical price data for the underlying has the potential to 'add value' in markets where traded option prices are unavailable or unreliable. The AC08 method, as with Stutzer's (1996) Canonical valuation, is unusual in that it does not require historical option price data. However when option price data is available and reliable, this data is clearly the best source of information on current market option pricing. Traded option prices contain information on several price-sensitive factors, including market frictions, expected option payoff and, in incomplete markets, the market price of risk. Conversely if traded option prices are unavailable or unreliable, a nonparametric method that relies solely on option price data will find it difficult to eliminate pricing bias. A nonparametric American option pricing method that incorporates, without being solely reliant upon, traded option price data has until now proved an elusive goal.

The method we present utilises the principle of relative entropy, in a similar manner to Stutzer (1996) and Alcock and Carmichael (2008). The validity of entropic pricing schemes has been demonstrated in a number of recent papers. Alcock and Gray (2005) establish Stutzer's (1996) Canonical method of valuing European options as an implementable pricing scheme by deriving a dynamic delta hedge - allowing for the formation of a nonparametric replicating portfolio. Fritelli (2000) establishes the link between entropic pricing algorithms and an investor's utility function. Gray and Newman (2005) demonstrate the accuracy and precision of Stutzer's Canonical method of valuing European options using simulated Heston data. Gray, Edwards, and Kalotay (2007) empirically demonstrate the appealing statistical properties of Canonical pricing using Australian S&P 200 index options.

Importantly, Gray et al. (2007) illustrate the substantial pricing improvements obtained by incorporating information from a single traded option price in the Canonical estimates of European option prices. The present article presents an extension to the American option pricing approach introduced by Alcock and Carmichael (2008) that incorporates a small number of traded option prices as additional price constraints. Unlike the AC08 method, the extension we present is capable of incorporating additional price-sensitive information embodied in current market prices, such as transactions costs, tax effects and other market imperfections along with associated market risk premia. After the estimation of the EMM an adaptation of the Monte-Carlo approach of Longstaff and Schwartz (2001) is utilized to include the early exercise feature of the American options in accordance with Alcock and Carmichael (2008).

The precision and accuracy of our extension is determined using simulation experiments. Using simulation experiments allows us to carefully control the testing of our method and to consequently determine the sensitivity of our extension to the additional option pricing constraints. The results are related to findings of Gray and Newman (2005) and Gray et al. (2007) who report that constrained canonical estimators show significant improvements in pricing European options. The performance of the canonical methods is compared to other common pricing approaches: a Black-Scholes model and a finite difference method (Crank-Nicolson), both using historic volatility. Furthermore, we investigate the set of option prices constraints generate optimal pricing outcomes. The tests also provide insight into the limitations of the different pricing approaches for American options.

The remainder of this article is structured as follows: Section 2 introduces the constrained canonical valuation for American options (CONAA) with an arbitrary number of additional constraints as an extension of the method of Alcock and Carmichael (2008) and provides theoretical and mathematical justification of this pricing approach. Section 3 contains a simulation analysis of CONAA in order to determine the optimal implementation of the CONAA methodology and to investigate the sensitivity to the parameters of the additional option price constraint in a Black-Scholes environment. The performance of CONAA is further compared to the method of Alcock and Carmichael (2008) and other common option pricing methods. Finally, Section 4 contains conclusions and suggestions for future research.

2 Nonparametric Pricing of American Options

2.1 Risk-Neutral Valuation Framework

Our nonparametric American option valuation methodology, like the entropic methods of Stutzer (1996) and Alcock and Carmichael (2008), prices options assuming that no arbitrage opportunities exist. We use the same discrete-time normalization method as Stutzer. Specifically, at each future time T (that is N equidistant periods of length $\Delta t = \frac{T-t}{N}$ from the current time t), the price process is discounted by the product of one-period risk-free interest rates up to that time. [1] Denoting the price of the asset by S_τ and dividend payment at time τ by D_τ, the equivalent martingale probabilities at time T, π^*_{T-t}, must therefore satisfy:

$$\begin{aligned} S_t &= \mathbb{E}_{\mathbb{Q}}\left[\frac{S_T + D_T + \sum_{n=1}^{N-1} D_{t+n\Delta t}\prod_{s=n}^{N-1}(1+r_{t+s\Delta t})}{\prod_{n=1}^{N}(1+r_{t+n\Delta t})}\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\frac{S_T + D_T + \sum_{n=1}^{N-1} D_{t+n\Delta t}\prod_{s=n}^{N-1}(1+r_{t+s\Delta t})}{\prod_{n=1}^{N}(1+r_{t+n\Delta t})}\frac{d\pi^*_{T-t}}{d\pi_{T-t}}\right], \end{aligned} \tag{1}$$

where $r_{t+n\Delta t}$ represents the Δt-period risk-free interest rate up to time $t+n\Delta t$ $\{\forall n = 1,\dots,N\}$, π_{T-t} denotes the actual (real-world) probability measure at time T and $\frac{d\pi^*_{T-t}}{d\pi_{T-t}}$ denotes the Radon-

[1] Whilst this analysis assumes the derivative security is written on a single underlying asset, the same methodology can be generalized for multiple prices processes.

Nykodym density of the martingale measure at time T.[2]

Provided it is possible to estimate the risk-neutral measure that satisfies the constraints in (1), the risk-neutral valuation of a derivative security can be obtained by using the EMM to compute the expected discounted value of the derivative security.

Our valuation methodology is comprised of three distinct parts. First, time series of historical asset returns and risk-free rates are used to construct empirical probability distributions of possible prices for the underlying asset at N exercise dates between the current time t and the expiration date T. This stage involves the formation of M sample stock price paths, each of which has a estimated probability $\hat{\pi}^{(i)}$ attached. Second, the principle of relative entropy is employed to transform the assumed real-world probability distribution π into an estimated risk-neutral probability distribution $\hat{\pi}^*$ of the unknown EMM that satisfies (1). The selection of a suitable risk-neutral probability distribution is further constrained by the requirement that a small sample (typically 1) of traded options are accurately priced. Gray and Newman (2005) and Gray et al. (2007) demonstrate that such a constraint can deliver significant accuracy improvements in the Canonical pricing of European options. Third, a weighted least-squares algorithm is implemented to estimate the optimal exercise boundary of the option. The American option is valued by computing the expected discounted payoff using the estimated risk-neutral probabilities, $\hat{\pi}^*$.

The remainder of this Section introduces the three components of our valuation methodology in more detail. For the sake of simplicity it is assumed that the derivative security is an American option whose value is contingent on a single, non-dividend paying underlying asset (stock). Our valuation methodology will price the option at the current time $t = t_0$, assuming the option expires at time T and exercise opportunities are available at times $\{t_1, t_2, \ldots, t_{N-1}, T\}$. In addition, it is also assumed that the risk-free rate r between times t and $T = t_N$ is constant. Finally, the current stock price is denoted by S_t and the stock price process is assumed to be a stationary ergodic Markov process.

[2] $\mathbb{E}_{\mathbb{Q}}[\cdot]$ and $\mathbb{E}_{\mathbb{P}}[\cdot]$ represent expectations derived under the martingale and actual (real-world) probability measures, respectively.

2.2 Empirical Distribution of Sample Paths

Our valuation methodology assesses whether it is optimal to exercise the American option at the $(N-1)$ exercise times denoted by $\tau = t_1, t_2, \ldots, t_{N-1}$, in addition to the expiration date T via a weighted least squares algorithm adapted from Longstaff and Schwartz (2001).

In order to implement this, a sufficient number of non-overlapping $\left(\frac{T-t}{N}\right)$-period returns are required in order to construct M empirical stock price paths. Specifically, each sample stock price path consists of $(N+1)$ elements beginning with the current observed stock price S_t and the stock price at each of the N exercise dates, concluding with the stock price at expiry: $\left\{S_t^{(i)}, S_{t_1}^{(i)}, t_{t_2}^{(i)}, \ldots, S_{t_{N-1}}^{(i)}, S_T^{(i)},\ \forall i = 1, \ldots, M\right\}$. The construction of each sample path, therefore, requires $N \times \left(\frac{T-t}{N}\right)$-period returns:

$$
\begin{aligned}
S_{t_1}^{(i)} &= S_t^{(i)} R_{t_1 - t}^{(i)} \\
S_{t_n}^{(i)} &= S_{t_{n-1}}^{(i)} R_{t_n - t_{n-1}}^{(i)} \qquad = S_t^{(i)} R_{t_n - t}^{(i)} \qquad n = 2, \ldots, N
\end{aligned} \tag{2}
$$

where $R_\tau^{(i)}$ represents the gross (price relative) τ-year return used in the estimation of sample stock path i, $\forall i = 1, \ldots, M$. Typically, the number of sample paths, M, will be much greater than the number of exercise times, N.

One advantage of estimating future stock prices using the stock's historical distribution is that the method is more likely to capture stylized features of the data, including skewness and leptokurtosis, compared to parametric valuation methods (Gray and Newman, 2005). Clearly a potential limitation of our valuation method is that the method may be quite data intensive. Error-metric analysis in Section **??** presents positive results suggesting that the data requirements may be significantly reduced by the use of an appropriate variance reduction mechanism.

2.3 Determination of the Risk-Neutral Distribution

Each possible stock price path constructed from the empirical distribution in (2) is initially assigned an equal real-world probability [3] $\hat{\pi}^{(i)} = \frac{1}{M}$, $\forall i = 1, \ldots, M$. As we are pricing American style options, we note that these probabilities are assigned to each sample stock price path, and hence to every exercise time of the option, and not simply to the stock price at expiry.

[3] This assumption reflects the fact that the sample stock price paths are all constructed using the same time series historical returns data. An attractive feature of all entropic pricing methods, including Stutzer (1996), Alcock and Carmichael (2008) and the current method, is that the pricer is able to incorporate prior beliefs about the relationship between the historical stock price paths and future stock price processess by adjusting the real-world probabilities, $\hat{\pi}$. This issue is beyond the scope of non-parametric pricing, and so is not explored in detail here.

Fundamental to our valuation methodology is the transformation of the estimated real-world probabilities $\hat{\pi}^{(i)}$ into estimated risk-neutral probabilities $\hat{\pi}^{*(i)}$ $(\forall i = 1, \ldots, M)$ that satisfy the martingale measure defined in (1). Formally, this implies that the risk-neutral (martingale) probabilities attached to the M sample stock price paths must be estimated subject to the constraints that the expected returns on the stock between time t and each of the N exercise dates $\{t_1, t_2, \ldots, t_{N-1}, T\}$ are equivalent to the constant risk-free rate[4] r. For each potential exercise time of the American option, the expected discounted return is set to unity. That is,

$$(3) \qquad E_{\pi^*}\left(\frac{R_{t_j-t}}{exp(r(t_j-t))}\right) = \sum_{i=1}^{M} \frac{R_{t_j-t}^{(i)}}{exp(r(t_j-t))}\pi^{*(i)} = 1,\ t_1, \ldots, t_N.$$

In addition to the N arbitrage constraints, we constrain the probabilities $\pi^{*(i)}$ such that $\tilde{N} - N$ options traded on the same underlying are correctly priced. As we shall examine later, there are several different choices available to implement this constraint. For example, when pricing an American call option, do we choose to constrain our risk-neutral probability estimates by ensuring that a European call option is priced correctly, or do we choose a European put constraint? Alternatively, is it better to use both or are American option constraints preferable? The effects of this choice on the accuracy of our method is examined in the following Sections. However, American options should not be used for these constraints, simply to avoid computational loops. If, for example, we choose to ensure that a European call option is correctly priced, then we implement our constraint in the following manner:

$$(4) \quad E_{\pi^*}\left(\frac{\max(S_{t_j^{obs}} R_{T_j^{obs}-t_j^{obs}} - X_j^{obs}, 0)}{exp(r(T_j^{obs}-t_j^{obs}))}\right) = \sum_{i=1}^{M} \frac{\max(S_{t_j^{obs}} R_{T_j^{obs}-t_j^{obs}}^{(i)} - X^{obs}, 0)}{exp(r(T_j^{obs}-t_j^{obs}))}\pi^{*(i)} = C_j^{obs},$$

where $T_j^{obs} \leq T$, $S_{t_j^{obs}}$, X_j^{obs} and C_j^{obs} are the expiration date, the underlying, strike and market price of the European option observed at the time $t_j^{obs} < t$, $j = N+1, \ldots, \tilde{N}$.

To obtain the optimal risk-neutral distribution π^* we solve the following constrained optimization by employing the relative entropy principle of information theory (cf. Alcock and Carmichael, 2008; Stutzer, 1996);

$$(5) \qquad \inf_{\pi^{*(i)} \geq 0, \sum \pi^{*(i)} = 1, i=1,\ldots,M} \left\{ \sum_{i=1}^{M} \pi^{*(i)} \ln\left(\frac{\pi^{*(i)}}{\hat{\pi}^{(i)}}\right) : \sum_{i=1}^{M} g_j^{(i)} \pi^{*(i)} = a_j \right\}.$$

[4] Continuous compounding is assumed.

where $\sum_{i=1}^{M} \pi^{*(i)} \ln\left(\frac{\pi^{*(i)}}{\widehat{\pi}^{(i)}}\right)$ is the Kullback-Leibler distance that is minimized subject to the constraints $\sum_{i=1}^{M} g_j^{(i)} \pi^{*(i)} = a_j, \quad j = 1 \dots \tilde{N}$ where

$$g_j = \left[\frac{R^{(1)}_{t_j - t}}{exp(r(t_j - t))}, \dots, \frac{R^{(M)}_{t_j - t}}{exp(r(t_j - t))}\right], \qquad a_j = 1, \tag{6}$$

for $j = 1, \dots, N$, and

$$g_j = \left[\frac{\max(S_{t_j^{obs}} R^{(1)}_{T_j^{obs} - t_j^{obs}} - X_j^{obs}, 0)}{exp(r(T_j^{obs} - t_j^{obs}))}, \dots, \frac{\max(S_{t_j^{obs}} R^{(M)}_{T_j^{obs} - t_j^{obs}} - X_j^{obs}, 0)}{exp(r(T_j^{obs} - t_j^{obs}))}\right], \qquad a_j = C_j^{obs}, \tag{7}$$

for $j = N+1, \dots, \tilde{N}$.

Following Ben-Tal's Duality Theorem (1985, page 269, Theorem 1), the risk-neutral distribution π^* that optimizes (5) is given [5] by

$$\pi^{*(i)} = \frac{\widehat{\pi}^{(i)} \exp \sum_{j=1}^{\tilde{N}} \lambda_j^* g_j^{(i)}}{\sum_{i=k}^{M} \widehat{\pi}^{(i)} \exp \sum_{j=1}^{\tilde{N}} \lambda_j^* g_j^{(k)}} = \frac{\exp \sum_{j=1}^{\tilde{N}} \lambda_j^* g_j^{(i)}}{\sum_{k=1}^{M} \exp \sum_{j=1}^{\tilde{N}} \lambda_j^* g_j^{(k)}}, \tag{8}$$

where the vector of Lagrange multipliers λ^* is given as

$$\lambda^* = \arg\min_{\lambda} \left\{ \sum_{i=1}^{M} \exp\left(\sum_{j=1}^{\tilde{N}} \lambda_j (g_j^{(i)} - a_j) \right) \right\}. \tag{9}$$

2.4 Weighted Least-Squares (WLS) Algorithm

The final step of our valuation methodology involves the determination of the optimal exercise strategy for the American option over the M sample paths to which we have assigned risk-neutral weights as above. To this end, we utilise Alcock and Carmichael's (2008) WLS adaptation of the Longstaff and Schwartz's (2001) least-squares method for Monte-Carlo pricing of American options. The LSM method estimates the conditional expected payoff of 'keeping the option alive' at each potential exercise point by regressing the "ex-post realized payoffs from continuation" on a set of basis functions of the values of the relevant state variables.

[5] A more detailed derivation is provided in Appendix A.

The LSM algorithm is a Monte-Carlo method utilizing sample paths simulated under the risk-neutral measure with each path contributing equally to the final valuation. Clearly this cannot hold in the context of a nonparametric pricing method where the stock price paths are observed under the real-world measure, $\mathbb{P}$ rather than simulated under the risk-neutral measure $\mathbb{Q}$. Alcock and Carmichael (2008) compensate for this by using a weighted least squares, rather than an ordinary least squares, methodology.

2.4.1 LSM modifications

In order to estimate the conditional expectation function at the k^{th} early exercise date at time t_k $\{\forall k = 1, \ldots, N-1\}$, a $[M^*_{t_k} \times M^*_{t_k}]$ diagonal matrix, $\mathbf{W}_{t_k}$, must be constructed, such that the leading diagonal represents the weights (risk-neutral probabilities) of the $M^*_{t_k}$ in-the-money sample paths at time t_k. The utilization of only those sample paths that are in-the-money is consistent with the methodology of Longstaff and Schwartz (2001, p. 123).

As presented in Longstaff and Schwartz (2001), let $\mathbf{d}^*_{t_k}$ represent the $[M^*_{t_k} \times 1]$ vector of discounted future cash flows of the $M^*_{t_k}$ in-the-money sample paths at time t_k. We seek to estimate the optimal exercise boundary function using a finite set of basis functions [6] at each stopping time, t_k. [7] Let $\mathbf{X}^*_{t_k}$ represent a $[M^*_{t_k} \times (B+1)]$ matrix whose rows evaluate the B basis functions (plus the intercept) at the stock prices [8] of the $M^*_{t_k}$ in-the-money sample paths at time t_k $\{L_0(S_{t_k}), L_1(S_{t_k}), \ldots, \; L_B(S_{t_k})\}$. Following Alcock and Carmichael (2008) and Stentoft (2004), we implement the WLS algorithm using $B = 2$ Legendre polynomial basis functions (plus the intercept):

$$L_0(\frac{S_{t_k}}{K}) = 1, \qquad L_1(\frac{S_{t_k}}{K}) = 2\left(\frac{S_{t_k}}{K}\right) - 1, \qquad L_2(\frac{S_{t_k}}{K}) = 6\left(\frac{S_{t_k}}{K}\right)^2 - 6\left(\frac{S_{t_k}}{K}\right) + 1.$$

Longstaff and Schwartz (2001) utilize an OLS regression at each early exercise opportunity t_k $\{\forall k = 1, \ldots, N-1\}$ to estimate the $[(B+1) \times 1]$ vector $\mathbf{c}_{t_k}$ of regression coefficients corresponding to the B basis function (plus the intercept) $\{c_{0,t_k}, c_{1,t_k}, \ldots, c_{B,t_k}\}$.

[6] Every quadratically integrable function on the Real plane can be represented as a linear combination of orthogonal basis functions. A finite combination of orthogonal basis functions yields an approximation to the quadratically integrable function.

[7] Note that the number of early exercise opportunities, N, can be large so that this boundary will closely approximate the continuous early exercise boundary.

[8] Both Longstaff and Schwartz (2001) and Stentoft (2004) advise that the stock prices which are input into the basis functions should be normalized by the strike price to avoid numerical scaling problems such as computational underflows. The results of unreported tests support this notion, and it is therefore recommended that this step should be performed when implementing this valuation methodology.

The estimator for $\hat{\mathbf{c}}_{t_k}$ can be derived in terms of a Method of Moments estimator:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\mathbf{X}_{t_k}^{*\prime}\epsilon\right] &= 0 \\
\mathbb{E}_{\mathbb{P}}\left[\mathbf{X}_{t_k}^{*\prime}\mathbf{d}_{t_k}^{*}\right] &= \mathbb{E}_{\mathbb{P}}\left[\mathbf{X}_{t_k}^{*\prime}\mathbf{X}_{t_k}^{*}\mathbf{c}_{t_k}\right] \\
\frac{1}{M_{t_k}^{*}}\sum_{i=1}^{M_{t_k}^{*}}\mathbf{X}_{t_k}^{*\prime}(\cdot,i)\mathbf{d}_{t_k}^{*}(i) &= \frac{1}{M_{t_k}^{*}}\sum_{i=1}^{M_{t_k}^{*}}\mathbf{X}_{t_k}^{*\prime}(\cdot,i)\mathbf{X}_{t_k}^{*}(i,\cdot)\mathbf{c}_{t_k} \\
\hat{\mathbf{c}}_{t_k} &= \left(\mathbf{X}_{t_k}^{*\prime}\mathbf{X}_{t_k}^{*}\right)^{-1}\mathbf{X}_{t_k}^{*\prime}\mathbf{d}_{t_k}^{*}, \qquad (10)
\end{aligned}$$

where $\epsilon = \{\epsilon_1, \ldots, \epsilon_{M_{t_k}^*}\}$ are the errors of regression. Under the risk-neutral measure, $\mathbb{Q}$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\mathbf{X}_{t_k}^{*\prime}\epsilon\right] &= 0 \\
\mathbb{E}_{\mathbb{Q}}\left[\mathbf{X}_{t_k}^{*\prime}\mathbf{d}_{t_k}^{*}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbf{X}_{t_k}^{*\prime}\mathbf{X}_{t_k}^{*}\mathbf{c}_{t_k}\right] \\
\sum_{i=1}^{M_{t_k}^{*}}\hat{\pi}^{*(i)}\mathbf{X}_{t_k}^{*\prime}(\cdot,i)\mathbf{d}_{t_k}^{*}(i) &= \sum_{i=1}^{M_{t_k}^{*}}\hat{\pi}^{*(i)}\mathbf{X}_{t_k}^{*\prime}(\cdot,i)\mathbf{X}_{t_k}^{*}(i,\cdot)\mathbf{c}_{t_k}.
\end{aligned}$$

Setting $\mathbf{W}_{t_k}$ a diagonal matrix with $\mathbf{W}_{t_k}(i,i) = M_{t_k}^{*}\hat{\pi}^{*(i)}$ yields

$$\frac{1}{M_{t_k}^{*}}\sum_{i=1}^{M_{t_k}^{*}}\mathbf{X}_{t_k}^{*\prime}(\cdot,i)\mathbf{W}_{t_k}(i,i)\mathbf{d}_{t_k}^{*}(i) = \frac{1}{M_{t_k}^{*}}\sum_{i=1}^{M_{t_k}^{*}}\mathbf{X}_{t_k}^{*\prime}(\cdot,i)\mathbf{W}_{t_k}(i,i)\mathbf{X}_{t_k}^{*}(i,\cdot)\mathbf{c}_{t_k},$$

and so the WLS estimator for $\mathbf{c}_{t_k}$ is given by

$$\hat{\mathbf{c}}_{t_k} = \left(\mathbf{X}_{t_k}^{*\prime}\mathbf{W}_{t_k}\mathbf{X}_{t_k}^{*}\right)^{-1}\mathbf{X}_{t_k}^{*\prime}\mathbf{W}_{t_k}\mathbf{d}_{t_k}^{*}. \qquad (11)$$

Hence we employ a WLS regression algorithm (11), rather than the OLS regression algorithm used by Longstaff and Schwartz (2001), in order to effect a change of measure from $\mathbb{P}$ to $\mathbb{Q}$. At each early exercise date $\{t_k, k = 1, 2, \ldots, N-1\}$, the $[M_{t_k}^* \times 1]$ vector containing the fitted continuation values for each of the $M_{t_k}^*$ in-the-money paths is given by $\mathbf{X}_{t_k}^*\hat{\mathbf{c}}_{t_k}$. Using this vector, the fitted continuation value is compared against the early-exercise cash flows to determine if early exercise is optimal for each of the $M_{t_k}^*$ in-the-money-sample paths at time t_k.[9]

In accordance with the WLS approach, we utilise a recursive procedure in which the optimal exercise strategy is determined for each of the M sample stock price paths. The final step in

[9] Continuation is assumed optimal for the $(M - M_{t_k}^*)$ at-the-/out-of- the-money sample stock price paths at time t_k.

our valuation methodology uses this optimal exercise strategy to price American options using the estimated martingale probabilities to compute the expected discounted payoff. In particular, our estimated price of an American put option $P_t(S_t)$ at time t with strike price K is given by:

$$P_t(S_t) = \sum_{i=1}^{M} \left(\frac{\max\left(K - S_t R^{(i)}_{z^{(i)}-t}, 0\right)}{\exp\left(r\left(z^{(i)} - t\right)\right)} \right) \hat{\pi}^{*(i)}. \tag{12}$$

where S_t is the actual current stock price and $z^{(i)}$ is the optimal exercise date of sample path i according to the WLS methodology (if exercise is indeed optimal at all). As previously discussed, exercise is limited to N dates only $\{\tau = t_1, t_2, \ldots, t_{N-1}, T\}$. Similarly, the price of an American call option $C_t(S_t)$ at time t with strike price K is given by:

$$C_t(S_t) = \sum_{i=1}^{M} \left(\frac{\max\left(S_t R^{(i)}_{z^{(i)}-t} - K, 0\right)}{\exp\left(r\left(z^{(i)} - t\right)\right)} \right) \hat{\pi}^{*(i)}. \tag{13}$$

3 Optimal Implementation and Error Analysis

We investigate the accuracy of our pricing method, (12) and (13), using a controlled simulation analysis. The performance of our method is benchmarked against the nonparametric pricing method of Alcock and Carmichael (2008), the numerical Crank-Nicolson price and the relevant Black-Scholes price, both calculated using historic volatility. Utilising simulation tests allows us to investigate the performance of our approach in a controlled environment that enables careful sensitivity analysis to be undertaken. We also examine the role of different option price constraints on the accuracy of our method, and ascertain the optimal combination of options to use as pricing constraints.

Determining a suitable option price constraint, (4), will often be dictated by market conditions. In the case that a choice of reliable option price constraints is available, determining the best subset for implementation may appear unclear. Heston (1993) suggests that all derivatives written on the same set of underlyings should attract the same market risk premium. So it might appear that we have an arbitrary choice of the constraining option within our pricing method. An arbitrary choice is not necessarily available, however. As already stated, utilising American option price constraints is not advisable for computational reasons. Also, any difference in informational content between European call and put options may generate a systematic difference in the pricing accuracy of our method. In Section 3.2 we investigate which subset of European put and call options, when used as additional option price constraints, provide the greatest accuracy improvements in our pricing approach. In Section 3.3 we examine whether the time to maturity of the option used as conditioning information affects the pricing accuracy of our method. In Section 3.4 we perform a comparative error-metric analysis of our method.

3.1 Methodology

In the following three subsections we utilise simulations resulting from Black-Scholes assumptions. We assume the underlying to be a single, non-dividend paying security that is traded in a frictionless market. We also assume the existence of a fixed, finite number of potential exercise opportunities $t_0, \ldots, t_N$.

To price an American option at time t with a given underlying price, strike price, and expiration date T with our method (CONAA) and the AC08 method we simulate M discrete asset price paths $\{S_{t_0}, \ldots, S_{t_N}\}, t_0 = t, T_N = T$ under a Geometric Brownian Motion [10] (GBM), given by

$$S_T = S_t \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma W_{(T-t)}\right]. \tag{14}$$

Our method estimates the discrete EMM, (8), given the constraints (3) and (4). The observed market prices required as additional constraints (4) are given by exact Black-Scholes prices of European call and put options. The volatility used for calculating the 'observed' Black-Scholes prices is the same as used in (14) as the European options are written on the same underlying asset. For the AC08 method, the estimation of the EMM constrained solely by (3).

The simulated stock price paths are also treated as 'historical' stock price paths in order to estimate the historical volatility of the underlying asset price. The historical volatility is required to calculate price estimates for the American options using the Black-Scholes formula (HBS) and the Crank-Nicolson (HCN) numerical technique (using a 750×750 grid). By doing so, we ensure that both the parametric pricing techniques use the same information as the two nonparametric methods.

The American price estimates are compared with 'true prices' given by the Black-Scholes price

[10] In the Black-Scholes model μ is the drift, σ is the constant asset price volatility, W_t is a standard Wiener process and S_0 is the initial asset price.

for the American call options, and the Crank-Nicolson (750×750 grid) price for American put options, both employing the true volatility of the underlying. Using 1000 estimated prices, we calculate mean absolute percentage error (MAPE) and mean square error (MSE) for each of the four pricing methods. We further decompose the MSE into variance and square bias and present the variance-MSE ratio.

3.2 Sensitivity to the Type of Option Price Constraint

In contrast to the AC08 approach, our method includes information on observed market option prices as additional constraints. As described in Section 2, it is possible to utilize call or put options or even more than one option price constraint. It is not only of interest to analyse which type of constraint is optimal to use but also whether two different price constraints lead to a better result. A comparison to the AC08 method will show whether including these extra constraints improves the performance of non-parametric pricing of American options.

We price American options, using a variety of parameter sets, with the AC08 method and three versions of our constrained valuation methodology (CONAA): (i) with a European call price as constraint (CONAA call), (ii) with a European put price as constraint (CONAA put) and (iii) with both a European call and put price as constraints (CONAA call & put). The parameters varied are the moneyness of the option to be priced and the moneyness of the option price constraints utilized in CONAA. Table 1 presents MSEs and MAPEs of the price estimates of the American options for each of the the four investigated methods. 2 contains bias and squared bias of the different methods, and 3 the variance expressed as a proportion of MSE. Figures 1, 2 and 3 show plots of MSE, MAPE and bias values in Table 1 and 2.

[Table 1 about here]

[Table 2 about here]

[Table 3 about here]

[Figure 1 about here]

[Figure 2 about here]

[Figure 3 about here]

Incorporating option price constraints generally reduces error, as measured by both MSE and MAPE. Irrespective of the type of option price constraint used, our price estimates have lower MSE and MAPE than the unconstrained method AC08 (Table 1). For pricing in-the-money options, less relative value is added by utilising an option constraint. For these options, the unconstrained method, AC08, produces a similar level of error as the constrained method with one option price constraint. Our method requires two given price constraints in order to outperform the AC08 approach for pricing in-the-money options.

In general, CONAA with both a call and put option constraint tends to demonstrate the best performance compared to the other three approaches. This is to be somewhat expected, as the algorithm has the advantage of two additional information sources. However, giving the method more than one price constraint can cause convergence problems. If the two option prices reflect contradictory information specific to each type of option, then the optimisation process in (9) and (8) will not yield a unique solution.

For pricing American call options, "CONAA call" and "CONAA call & put" show similar levels of accuracy. Using a put option constraint to price an American call option appears to generate the greatest error. Similarly, "CONAA put" and "CONAA call & put" generate similar levels of accuracy for pricing American put options. Using a call option constraint to price an American put option leads to worse results than having no option price constraint for in-the-money puts.

The performance of the constrained method seems to be more dependent on the moneyness of the option priced than on the moneyness of the option price constraint. The results of this simulation analysis indicate no clear preference for the moneyness of the option price constraint. More generally, Tables 1 and 2 suggest that the mean square error of all methods is most severe for in-the-money call and put options, where the pricing errors for in-the-money calls tend to be smaller than for in-the-money put options. However, the price relative effect of these errors (MAPE) is small due to high values of in-the-money options. Overall it appears optimal to provide information on both European call and put prices as constraints, so long as this does not lead to convergence problems. Otherwise American call options are best priced with a European call option constraint, and American put options with a European put option constraint.

The bias and the squared bias values (Table 2) confirm the improved performance of our method for at-the-money and out-of-the-money options. Furthermore, our methodology is found to be essentially unbiased (Table 3), and so can be considered a consistent estimator of the true American option price. This result is similar to that found by Alcock and Carmichael (2008). That is, these results confirm that the addition of option price constraints does not significantly affect the bias of this class of pricing estimators. The small amount of bias that is present might be mitigated even further by averaging the results of the constrained and unconstrained methodologies. The AC08 price estimates have a very slight negative bias for all priced American options except American call options with moneyness $\frac{S_0}{K} = 1.2$. CONAA price estimates generally show a very slight positive bias for American call options and out-of-the-money American put options, and a negative bias for American put options that are in-the-money or at-the-money. In many cases, the bias of these two methods will cancel with each other.

The bias-variance decomposition presented by the ratio of variance and MSE in Table 3 reveals that the variance usually contributes a high proportion of the MSE for all entropic estimators. The average variance-MSE ratio for American call options is 75% and the average variance-MSE ratio for American put options is 61% where the value for American put option would increase to 73% if the American put options with moneyness 1.2 are left out. Considering only the constrained method the average variance proportion of MSE is 77% for American calls, and 62% (74%) for American puts (without moneyness 1.2) irrespective the type of option price constraint utilized. Altogether, this shows that the pricing errors are to a major extent caused by the variance of the price estimates and to a lesser extent by the bias. These results strongly indicate that the application of variance reduction mechanisms should lead to substantial improvements in terms of pricing accuracy.

3.3 Sensitivity to the Time to Maturity of the Option Price Constraint

The time to maturity $T_{obs} - t_{obs}$ of the observed options is not required to coincide with the time to maturity $T - t$ of the option that is priced [11]. This section analyses the sensitivity of our method to the time-to-maturity of the option price constraint. Following the results of Section 3.2 we price American call options with a European call option as the additional constraint and American put options with a European put option constraint. Each American call and put option price is calculated using $L = 1000$ different simulation runs. Table 4(a) presents the MSE, Table

[11] The time to maturity of the observed option $T_{obs} - t_{obs}$ must be less than or equal to $T - t$. This is necessary as the last date covered in the simulated asset price paths is T.

4(b) the MAPE, and Table 4(c) the squared bias of the CONAA estimates with the different option price constraints. Table 4(d) shows the variance-MSE ratio obtained by bias-variance decomposition. Figures 4, 5 and 6 show plots of the MSE, MAPE and squared bias values in Table 4.

[Table 4 about here]

[Figure 4 about here]

[Figure 5 about here]

[Figure 6 about here]

The values of MSE, MAPE, and squared bias show that the time to maturity of the "observed" option does not have a major influence on the accuracy of CONAA. Nevertheless, CONAA using a option price constraint with $T_{obs} - t_{obs} = T - t = \frac{1}{2}$ shows slightly lower values of MSE, MAPE and squared bias for options that are in-the-money and out-of-the-money, but the differences to CONAA using "observed" options with a shorter time to maturity are unlikely to be significant. Using an option price constraint with $T_{obs} - t_{obs} = \frac{1}{4}$ generates slightly higher MSE, MAPE and squared bias than the option price constraints with the longer or shorter time to maturity. Again, these differences are unlikely to be significant.

As in Section 3.2 the accuracy of CONAA is largely unaffected by the moneyness of the option price constraint, and there is no clear preference for the moneyness of the option constraint. In-the-money call and put options are the most mispriced in terms of MSE, while MAPEs indicate that this mispricing is small relative to the price magnitude, due to the high option values of in-the-money options. CONAA price estimates for call options are generally more accurate through all maturities, than price estimates for put options. Overall the time to maturity of the "observed" option should ideally match the time to maturity of the option that is priced, but a difference in the time to maturities does not lead to a significant reduction in accuracy.

These results are not surprising given that we are simulating an ideal market. If price-sensitive information, such as tax rate, has a non-constant effect on traded option prices then pricing accuracy will be maximised by choosing the option price constraint to have the same time to maturity as the option being priced.

3.4 Evaluation of CONAA Against Other Methods

Simulation experiments conducted by Gray and Newman (2005) show that pricing accuracy of canonical valuation can be significantly improved for European options by including an additional price constraint. Alcock and Carmichael (2008) illustrate the accuracy of the AC08 method when compared to HBS. This section compares the pricing accuracy of our constrained method for American-style options, along with the AC08 method, to the historical volatility based valuation approaches HBS and HCN. Recall that only American call options can be priced under Black-Scholes if the underlying asset does not pay dividends (Merton, 1973). In this case, the price of American call options coincides with the price of the corresponding European call option [12] . We also apply the Black-Scholes model for American put options to compare our method to this still widely used option pricing model. [13]

The statistical properties of the pricing errors (MSE, MAPE, bias, squared bias, variance-MSE ratio) under each of these four methods are compared to evaluate their relative performance. The results of the error analysis are presented in Table 5 (MSE, MAPE, bias) and 6 (squared bias, variance-MSE ratio). Figures 7, 8, 9 and 10 visualize the MSE, MAPE, bias and squared bias values in Table 5 and 6.

[Table 5 about here]

[Table 6 about here]

[Figure 7 about here]

[Figure 8 about here]

[Figure 9 about here]

[Figure 10 about here]

For in-the-money call options HBS and HCN price estimates show higher accuracy than the

[12] That is the European call option with the same underlying asset price, the same strike price and same time to maturity as the American option.

[13] As the Black-Scholes option prices increase with an increasing volatility, the volatility used for European options is often simply “marked up” to price an American option.

canonical estimators. The reason for the superior performance of these historical volatility based methods is that the 'true' price of the American call is given by the Black-Scholes European call price as the asset is assumed to be non-dividend paying. Moreover, it is well known that the valuation of in-the-money call options, using common pricing methods like Black-Scholes and finite difference methods, is straightforward.

Our constrained valuation usually outperforms AC08 for out-of-the-money and at-the-money American call options, while AC08 outperforms CONAA for deep in-the-money calls. Moreover, CONAA is clearly superior to HBS and HCN for out-of-the-money and at-the-money American call options where AC08 shows a similar or only slight improvement in pricing accuracy. This result holds independently of the criterion used for comparison - MSE (Table 5 (a)), bias (Table 5 (c)) and squared bias (Table 5 (d)). The high MAPEs (Table 5 (b)) for these options can partially be ascribed to very small option prices. CONAA and AC08 are still essentially unbiased, while HBS and HCN consistently underprice the American call options. This difference is particularly noticeable for out-of-the money call options with longer maturity.

The results for American put options are slightly different than for American call options. CONAA again outperforms AC08 for out-of-the-money and at-the-money options in terms of MSE. The CONAA and AC08 methods are clearly superior to the HBS estimator in terms of MSE, MAPE, bias and squared bias. The HBS estimator regularly underprices American put options. This result is expected as the Black-Scholes price does not incorporate the value of the early exercise opportunity of the American put option. For in-the-money options HCN generates the best results, largely because the 'true' American put options price are given by a Crank-Nicolson approximation. However, for out-of-the-money and at-the-money options, the CONAA and AC08 estimators are superior to HCN. As for American call options, HCN underprices these options for all levels of moneyness and time to maturity.

A final comment relates to the results of the bias-variance decomposition of the different tested methods, presented in Table 6(b). Whilst CONAA/AC08 have an average variance proportion of MSE of 72%/75% for American call options and 75%/77% for American put options, the common pricing methods HBS/ HCN have an average variance proportion of MSE of 45%/46% for American call options and 16%/33% for American put options. This presents a major advantage of the nonparametric estimators over the common pricing methods as it is much easier to decrease the variance of price estimates of a method than the bias. Thus, the application of variance reduction mechanisms can be expected to lead to a reduction in MSE for the nonparametric methods and thus to a better performance in comparison to HBS and HCN.

4 Conclusions

In this article we introduce and test a nonparametric method for valuing American style options. This method is an extension of the method presented by Alcock and Carmichael (2008) that uses time series of historical asset prices to estimate the price of American options written on that asset. Our extension incorporates conditioning information from a small set of observed option prices. Ensuring that this set of options is correctly priced allows our model to incorporate price sensitive information resulting from market imperfections.

We evaluate the pricing accuracy and precision of our method in a controlled Black-Scholes simulation environment. We find that our method is both accurate and precise and is essentially a consistent estimator of the true American option price. The inclusion of conditioning information generates significant reductions in variance, without compromising bias. Furthermore our method is more accurate and precise than either Black Scholes or Crank-Nicolson estimators that use historical volatility.

We also examine the optimal implementation of our method. We find that American put options should be priced with a European put option constraint, and similarly, American call options with a European call option constraint. Having more than one option as additional constraint can cause convergence problems if the different option prices indicate a different market price of risk. It is optimal if the maturity of the option price constraint coincides with the maturity of the American option to be priced. However, in general, the time to maturity of the option price constraint has little effect on the accuracy of our methodology.

With the development and extensive testing of our and other methods this study makes an important contribution to the growing literature in the field of nonparametric American option pricing. However, there is room for further improvements and refinements. Neither our method, nor the AC08 method have been tested using market data. While simulation studies are important to identify the characteristics of the method as opposed to characteristics of the market, market-based tests are nevertheless vital to confirm the validity of these methodologies. Another avenue for future research is the development of a dynamic delta hedging formula, similar to that presented by Alcock and Gray (2005) in the context of Canonical pricing of European put and call options. Finally, an investigation into appropriate variance reduction methods is recommended for the entropic methods as all results of the simulation analysis indicate that the variance usually comprises the majority of the MSE for almost all levels of moneyness and time to maturity. These appear to be worthy avenues for future research efforts.

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A Incorporating Option Price Constraints in the Risk-Neutral Distribution

To solve the constrained optimisation (5), we utilize a simple Lagrange multiplier approach in combination with the Duality Theorem and Lemma 1 in Ben-Tal (1985, page 268-269). Here we derive the vector of Lagrange multipliers that needs to be determined in order to calculate the EMM according to (8). We begin with the derivation of the Lagrangian function $L(\pi^*, \widehat{\pi})$ of (5),

$$\text{(A.1)} \qquad L(\pi^*, \widehat{\pi}) = \sum_{i=1}^{M} \pi^{*(i)} \ln\left(\frac{\pi^{*(i)}}{\widehat{\pi}^{(i)}}\right) - \sum_{j=1}^{\tilde{N}} \lambda_j \left(\sum_{i=1}^{M} g_j^{(i)} \pi^{*(i)} - a_j\right).$$

As $\sum\limits_{i=1}^{M} \pi^{*(i)} = 1$, (A.1) can be rearranged the following way

$$\text{(A.2)} \qquad L(\pi^*, \widehat{\pi}) = \sum_{i=1}^{M} \pi^{*(i)} \ln\left(\frac{\pi^{*(i)}}{\widehat{\pi}^{(i)}}\right) - \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{M} \lambda_j (g_j^{(i)} - a_j) \pi^{*(i)}.$$

Due to the finiteness of the sums the Σ signs in the second part of the equation can be switched

$$\text{(A.3)} \qquad L(\pi^*, \widehat{\pi}) = \sum_{i=1}^{M} \pi^{*(i)} \ln\left(\frac{\pi^{*(i)}}{\widehat{\pi}^{(i)}}\right) - \sum_{i=1}^{M} \sum_{j=1}^{\tilde{N}} \lambda_j (g_j^{(i)} - a_j) \pi^{*(i)},$$

and with algebraic manipulation the formula reduces to

$$\text{(A.4)} \qquad L(\pi^*, \widehat{\pi}) = \sum_{i=1}^{M} \pi^{*(i)} \ln\left(\frac{\pi^{*(i)}}{\widehat{\pi}^{(i)} \exp\left(\sum\limits_{j=1}^{\tilde{N}} \lambda_j (g_j^{(i)} - a_j)\right)}\right).$$

Setting $c(t) = \widehat{\pi}^{(i)} \exp\left(\sum\limits_{j=1}^{\tilde{N}} \lambda_j (g_j^{(i)} - a_j)\right)$ and applying Lemma 1 of Ben-Tal (1985, page 268), the dual objective function is determined as

$$\text{(A.5)} \quad H(\lambda) = \inf_{\pi^{*(i)} \geq 0, \sum \pi^{*(i)} = 1, i=1,\ldots,M} \{L(\pi^*, \widehat{\pi})\} = -\ln\left(\sum_{i=1}^{M} \widehat{\pi}^{(i)} \exp\left(\sum_{j=1}^{\tilde{N}} \lambda_j (g_j^{(i)} - a_j)\right)\right).$$

The risk-neutral distribution π^* that optimizes (5) is determined by finding the vector of Lagrange multipliers λ^* that maximizes $H(\lambda)$:

$$\text{(A.6)} \qquad \lambda^* = \arg\max_{\lambda} H(\lambda) = \arg\max_{\lambda} \left\{ -\ln\left(\sum_{i=1}^{M} \widehat{\pi}^{(i)} \exp\left(\sum_{j=1}^{\tilde{N}} \lambda_j (g_j^{(i)} - a_j)\right)\right)\right\}.$$

As $\widehat{\pi}^{(i)} = \frac{1}{M}$, $\forall i = 1, \ldots, M$ and as $\ln(x)$ is a strictly monotonic increasing function, (A.6) reduces to

$$\lambda^* = \arg\min_{\lambda} \left\{ \sum_{i=1}^{M} \exp\left(\sum_{j=1}^{\tilde{N}} \lambda_j (g_j^{(i)} - a_j) \right) \right\}. \tag{A.7}$$

Tables and Figures

Price Constr.	Method	MSE American Call S_0/K 0.8	0.9	1.0	1.1	1.2	MSE American Put K/S_0 0.8	0.9	1.0	1.1	1.2
$S_{obs} = 39$	**CONAA call (c)**	0.0006	0.0009	0.0006	0.0022	0.0039	0.0003	0.0007	0.0026	0.0050	0.0078
$K_{obs} = 39$	**CONAA put (p)**	0.0007	0.0015	0.0027	0.0030	0.0026	0.0004	0.0006	0.0014	0.0052	0.0063
	CONAA c&p	0.0006	0.0009	0.0006	0.0010	0.0009	0.0003	0.0006	0.0014	0.0040	0.0051
	AC08	0.0010	0.0039	0.0057	0.0027	0.0020	0.0004	0.0017	0.0038	0.0037	0.0065
$S_{obs} = 39$	**CONAA call (c)**	0.0007	0.0023	0.0014	0.0008	0.0028	0.0003	0.0013	0.0042	0.0041	0.0077
$K_{obs} = 35.45$	**CONAA put (p)**	0.0007	0.0018	0.0025	0.0034	0.0034	0.0003	0.0005	0.0029	0.0051	0.0065
	CONAA c&p	0.0007	0.0015	0.0010	0.0008	0.0009	0.0003	0.0005	0.0024	0.0039	0.0052
	AC08	0.0009	0.0040	0.0052	0.0030	0.0019	0.0004	0.0016	0.0039	0.0037	0.0067
$S_{obs} = 39$	**CONAA call (c)**	0.0005	0.0004	0.0015	0.0023	0.0029	0.0003	0.0009	0.0018	0.0049	0.0081
$K_{obs} = 43.33$	**CONAA put (p)**	0.0008	0.0019	0.0030	0.0030	0.0021	0.0003	0.0008	0.0013	0.0040	0.0057
	CONAA c&p	0.0005	0.0004	0.0013	0.0017	0.0009	0.0003	0.0008	0.0012	0.0037	0.0053
	AC08	0.0009	0.0040	0.0051	0.0030	0.00020	0.0004	0.0017	0.0035	0.0040	0.0066
optimal	**CONAA call (c)**	0.0002	0.0004	0.0006	0.0008	0.0011	0.0004	0.0013	0.0027	0.0048	0.0072
	CONAA put (p)	0.0007	0.0017	0.0026	0.0034	0.0033	0.0002	0.0005	0.0014	0.0038	0.0048
	CONAA c&p	0.0002	0.0004	0.0006	0.0007	0.0010	0.0002	0.0005	0.0013	0.0034	0.0054
	AC08	0.0009	0.0041	0.0053	0.0029	0.0020	0.0004	0.0016	0.0037	0.0037	0.0066

(a) Mean Squared Error (MSE)

Price Constr.	Method	MAPE American Call S_0/K 0.8	0.9	1.0	1.1	1.2	MAPE American Put K/S_0 0.8	0.9	1.0	1.1	1.2
$S_{obs} = 39$	**CONAA call (c)**	0.0768	0.0218	0.0064	0.0072	0.0064	0.0868	0.0383	0.0229	0.0124	0.0082
$K_{obs} = 39$	**CONAA put (p)**	0.0847	0.0280	0.0141	0.0081	0.0052	0.0988	0.0340	0.0167	0.0124	0.0073
	CONAA c&p	0.0774	0.0217	0.0063	0.0046	0.0030	0.0961	0.0335	0.0165	0.0111	0.0066
	AC08	0.1073	0.0469	0.0213	0.0078	0.0045	0.1072	0.0591	0.0284	0.0103	0.0075
$S_{obs} = 39$	**CONAA call (c)**	0.0888	0.0348	0.0103	0.0040	0.0057	0.0881	0.0519	0.0302	0.0111	0.0082
$K_{obs} = 35.45$	**CONAA put (p)**	0.0854	0.0302	0.0136	0.0088	0.0059	0.0969	0.0313	0.0241	0.0123	0.0074
	CONAA c&p	0.0838	0.0277	0.0087	0.0038	0.0028	0.0949	0.0316	0.0222	0.0108	0.0067
	AC08	0.1023	0.0479	0.0204	0.0083	0.0042	0.1071	0.0570	0.0291	0.0103	0.0076
$S_{obs} = 39$	**CONAA call (c)**	0.0697	0.0148	0.0107	0.0073	0.0054	0.0916	0.0411	0.0192	0.0121	0.0085
$K_{obs} = 43.33$	**CONAA put (p)**	0.0910	0.0312	0.0150	0.0082	0.0047	0.0927	0.0382	0.0159	0.0111	0.0070
	CONAA c&p	0.0734	0.0148	0.0100	0.0062	0.0029	0.0918	0.0386	0.0154	0.0104	0.0067
	AC08	0.1021	0.0467	0.0203	0.0083	0.0044	0.1017	0.0584	0.0271	0.0108	0.0076
optimal	**CONAA call (c)**	0.0469	0.0136	0.0062	0.0040	0.0031	0.1014	0.0503	0.0232	0.0120	0.0080
	CONAA put (p)	0.0856	0.0292	0.0140	0.0088	0.0058	0.0832	0.0322	0.0165	0.0105	0.0064
	CONAA c&p	0.0461	0.0137	0.0062	0.0037	0.0029	0.0831	0.0320	0.0162	0.0100	0.0069
	AC08	0.1000	0.0476	0.0206	0.0080	0.0044	0.1028	0.0588	0.0278	0.0105	0.0076

(b) Mean Absolute Percentage Error (MAPE)

Table 1

Mean squared error (MSE) and mean absolute percentage error (MAPE) of the CONAA and AC0808 estimates relative to the Crank-Nicolson (750×750 grid) price for American put options and the Black-Scholes price for American call options, each determined with the parameters $S_0 = 40$, $T - t = 0.5$, $r = 0.06$ and $\sigma = 0.2$. The number of conducted price estimations is 1000. The asset is assumed to be non-dividend paying. Each cell presents the MSE/MAPE for an option with a certain moneyness that is priced with one of the four compared methods, where the parameters S_{obs} and K_{obs} of the option price constraint chosen for CONAA are given in the very left column. These parameters together with $T_{obs} - t_{obs} = 0.5$, $r = 0.06$, $\sigma = 0.2$ determine the constraining option price(s) as European call and put prices with the Black-Scholes formulas. The "optimal" price constraint is given as the Black-Scholes European call and put price for asset and strike price of the option that is priced, and $T - t = 0.5$, $r = 0.06$ and $\sigma = 0.2$. The addition to CONAA in column two tells which option type is used as constraint.

Price Constr.	Method	Bias American Call S_0/K					Bias American Put K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
$S_{obs} = 39$	**CONAA call (c)**	-0.0028	-0.0007	0.0149	0.0308	0.0446	0.0010	-0.0075	-0.0314	-0.0502	-0.0821
$K_{obs} = 39$	**CONAA put (p)**	0.0051	0.0168	0.0281	0.0211	0.0191	0.0100	0.0127	-0.0062	-0.0488	-0.0722
	CONAA c&p	0.0017	0.0073	0.0144	0.0080	0.0055	0.0093	0.0117	-0.0055	-0.0410	-0.0657
	AC08	-0.0192	-0.0434	-0.0484	-0.0170	0.0110	-0.0074	-0.0260	-0.0448	-0.0325	-0.0745
$S_{obs} = 39$	**CONAA call (c)**	-0.0113	-0.0278	-0.0162	0.0166	0.0422	-0.0050	-0.0223	-0.0476	-0.0403	-0.0822
$K_{obs} = 35.45$	**CONAA put (p)**	-0.0016	0.0001	0.0192	0.0391	0.0335	0.0137	0.0126	-0.0312	-0.0474	-0.0741
	CONAA c&p	-0.0044	-0.0106	0.0006	0.0160	0.0136	0.0133	0.0130	-0.0259	-0.0369	-0.0664
	AC08	-0.0169	-0.0448	-0.0462	-0.0165	0.0102	-0.0072	-0.0243	-0.0457	-0.0296	-0.0762
$S_{obs} = 39$	**CONAA call (c)**	0.0098	0.0145	0.0072	0.0147	0.0277	0.0000	-0.0082	-0.0180	-0.0442	-0.0845
$K_{obs} = 43.33$	**CONAA put (p)**	0.0150	0.0205	0.0054	-0.0019	0.0062	0.0035	0.0014	-0.0033	-0.0312	-0.0693
	CONAA c&p	0.0117	0.0144	-0.0046	-0.0106	-0.0020	0.0030	0.0006	-0.0034	-0.0293	-0.0672
	AC08	-0.0179	-0.0418	-0.0431	-0.0197	0.0113	-0.0072	-0.0254	-0.0407	-0.0338	-0.0756
optimal	**CONAA call (c)**	0.0111	0.0138	0.0137	0.0156	0.0194	-0.0087	-0.0216	-0.0290	-0.0434	-0.0796
	CONAA put (p)	0.0056	0.0215	0.0283	0.0368	0.0379	0.0124	0.0130	-0.0058	-0.0319	-0.0643
	CONAA c&p	0.0107	0.0134	0.0137	0.0148	0.0179	0.0123	0.0130	-0.0037	-0.0271	-0.0685
	AC08	-0.0173	-0.0436	-0.0455	-0.0195	0.0105	-0.0060	-0.0247	-0.0424	-0.0312	-0.0756

(a) Bias

Price Constr.	Method	Squared Bias American Call S_0/K					Squared Bias American Put K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
$S_{obs} = 39$	**CONAA call (c)**	0.0000	0.0000	0.0002	0.0010	0.0020	0.0000	0.0001	0.0010	0.0025	0.0067
$K_{obs} = 39$	**CONAA put (p)**	0.0000	0.0003	0.0008	0.0005	0.0004	0.0001	0.0002	0.0000	0.0024	0.0052
	CONAA c&p	0.0000	0.0001	0.0002	0.0001	0.0000	0.0001	0.0001	0.0000	0.0017	0.0043
	AC08	0.0004	0.0019	0.0023	0.0003	0.0001	0.0001	0.0007	0.0020	0.0011	0.0055
$S_{obs} = 39$	**CONAA call (c)**	0.0001	0.0008	0.0003	0.0003	0.0018	0.0000	0.0005	0.0023	0.0016	0.0068
$K_{obs} = 35.45$	**CONAA put (p)**	0.0000	0.0000	0.0004	0.0015	0.0011	0.0002	0.0002	0.0010	0.0023	0.0055
	CONAA c&p	0.0000	0.0001	0.0000	0.0003	0.0002	0.0002	0.0002	0.0007	0.0014	0.0044
	AC08	0.0003	0.0020	0.0021	0.0003	0.0001	0.0001	0.0006	0.0021	0.0009	0.0058
$S_{obs} = 39$	**CONAA call (c)**	0.0001	0.0002	0.0001	0.0002	0.0008	0.0000	0.0001	0.0003	0.0020	0.0071
$K_{obs} = 43.33$	**CONAA put (p)**	0.0002	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010	0.0048
	CONAA c&p	0.0001	0.0002	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0009	0.0045
	AC08	0.0003	0.0018	0.0019	0.0004	0.0001	0.0001	0.0007	0.0017	0.0012	0.0057
optimal	**CONAA call (c)**	0.0001	0.0002	0.0002	0.0002	0.0004	0.0001	0.0005	0.0008	0.0019	0.0063
	CONAA put (p)	0.0000	0.0005	0.0008	0.0014	0.0014	0.0002	0.0002	0.0000	0.0010	0.0041
	CONAA c&p	0.0001	0.0002	0.0002	0.0002	0.0003	0.0002	0.0002	0.0000	0.0007	0.0047
	AC08	0.0003	0.0019	0.0021	0.0004	0.0001	0.0000	0.0006	0.0018	0.0010	0.0057

(b) Squared Bias

Table 2

Bias and squared bias of the CONAA and AC08 estimates relative to the Crank-Nicolson (750×750 grid) price for American put options and the Black-Scholes price for American call options, each determined with the parameters $S_0 = 40$, $T - t = 0.5$, $r = 0.06$ and $\sigma = 0.2$. The number of conducted price estimations is 1000. The asset is assumed to be non-dividend paying. Each cell presents the bias/squared bias for an option with a certain moneyness that is priced with one of the four compared methods, where the parameters S_{obs} and K_{obs} of the option price constraint chosen for CONAA are given in the very left column. These parameters together with $T_{obs} - t_{obs} = 0.5$, $r = 0.06$, $\sigma = 0.2$ determine the constraining option price(s) as European call and put prices with the Black-Scholes formulas. The "optimal" price constraint is given as the Black-Scholes European call and put price for asset and strike price of the option that is priced, and $T - t = 0.5$, $r = 0.06$ and $\sigma = 0.2$. The addition to CONAA in column two tells which option type is used as constraint.

Price Constr.	Method	$Variance/MSE$ **American Call** S_0/K					$Variance/MSE$ **American Put** K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
$S_{obs} = 39$	**CONAA call (c)**	0.9863	0.9994	0.6357	0.5732	0.4864	0.9962	0.9252	0.6261	0.4951	0.1313
$K_{obs} = 39$	**CONAA put (p)**	0.9623	0.8114	0.7024	0.8494	0.8614	0.7183	0.7345	0.9730	0.5424	0.1668
	CONAA c&p	0.9948	0.9401	0.6402	0.9350	0.9670	0.7414	0.7647	0.9780	0.5827	0.1538
	AC08	0.6432	0.5213	0.5911	0.8947	0.9393	0.8649	0.6088	0.4771	0.7140	0.1420
$S_{obs} = 39$	**CONAA call (c)**	0.8243	0.6609	0.8154	0.6718	0.3697	0.9139	0.6249	0.4662	0.6038	0.1243
$K_{obs} = 35.45$	**CONAA put (p)**	0.9960	1.0000	0.8551	0.5533	0.6672	0.4302	0.6898	0.6586	0.5580	0.1510
	CONAA c&p	0.9698	0.9254	0.9996	0.6636	0.7910	0.4428	0.6784	0.7233	0.6491	0.1594
	AC08	0.6946	0.5045	0.5868	0.9093	0.9449	0.8742	0.6354	0.4695	0.7615	0.1336
$S_{obs} = 39$	**CONAA call (c)**	0.7900	0.4834	0.9651	0.9066	0.7382	1.0000	0.9223	0.8245	0.6032	0.1213
$K_{obs} = 43.33$	**CONAA put (p)**	0.7132	0.7727	0.9904	0.9988	0.9820	0.9637	0.9972	0.9914	0.7587	0.1617
	CONAA c&p	0.7282	0.4985	0.9834	0.9336	0.9956	0.9714	0.9996	0.9905	0.7676	0.1483
	AC08	0.6623	0.5661	0.6372	0.8712	0.9359	0.8595	0.6095	0.5321	0.7120	0.1347
optimal	**CONAA call (c)**	0.3793	0.4662	0.6708	0.7068	0.6447	0.7978	0.6390	0.6842	0.6062	0.1198
	CONAA put (p)	0.9529	0.7229	0.6870	0.6017	0.5670	0.3386	0.6762	0.9755	0.7346	0.1390
	CONAA c&p	0.4114	0.5137	0.6640	0.7015	0.6665	0.3427	0.6766	0.9894	0.7808	0.1378
	AC08	0.6693	0.5356	0.6053	0.8689	0.9452	0.9041	0.6227	0.5114	0.7376	0.1288

Variance-MSE Ratio

Table 3

Bias-variance decomposition: variance-MSE ratio for the CONAA and AC08 estimates. Variance is determined according as difference between MSE and squared bias. MSEs are taken over from Table 1, values for the squared bias from Table 2. Each cell presents the Variance-MSE-ratio for an option with a certain moneyness that is priced with one of the four compared methods (in 1000 runs), where the parameters S_{obs} and K_{obs} of the option price constraint chosen for CONAA are given in the very left column. These parameters together with $T_{obs} - t_{obs} = 0.5$, $r = 0.06$, $\sigma = 0.2$ determine the constraining option price(s) as European call and put prices with the Black-Scholes formulas. The "optimal" price constraint is given as the Black-Scholes European call and put price for asset and strike price of the option that is priced, and $T - t = 0.5$, $r = 0.06$ and $\sigma = 0.2$. The addition to CONAA in column two tells which option type is used as constraint.

Price Constraint		MSE American Call					MSE American Put				
S_{obs}, K_{obs}	$T_{obs} - t_{obs}$	S_0/K					K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
$S_{obs} = 39$	**1/13**	0.0015	0.0036	0.0039	0.0025	0.0030	0.0003	0.0011	0.0026	0.0028	0.0038
$K_{obs} = 39$	**1/4**	0.0017	0.0044	0.0048	0.0025	0.0020	0.0003	0.0011	0.0030	0.0039	0.0043
	1/2	0.0016	0.0039	0.0040	0.0029	0.0026	0.0003	0.0013	0.0033	0.0033	0.0041
$S_{obs} = 39$	**1/13**	0.0012	0.0025	0.0028	0.0023	0.0031	0.0003	0.0008	0.0017	0.0047	0.0048
$K_{obs} = 36$	**1/4**	0.0016	0.0036	0.0041	0.0022	0.0028	0.0003	0.0007	0.0023	0.0050	0.0050
	1/2	0.0010	0.0024	0.0032	0.0024	0.0027	0.0003	0.0010	0.0021	0.0029	0.0042
$S_{obs} = 39$	**1/13**	0.0008	0.0008	0.0006	0.0023	0.0040	0.0003	0.0006	0.0013	0.0051	0.0053
$K_{obs} = 43$	**1/4**	0.0012	0.0021	0.0012	0.0010	0.0033	0.0003	0.0005	0.0024	0.0055	0.0051
	1/2	0.0005	0.0004	0.0014	0.0022	0.0028	0.0003	0.0007	0.0013	0.0038	0.0044

(a) Mean Squared Error (MSE)

Price Constraint		MAPE American Call					MAPE American Put				
S_{obs}, K_{obs}	$T_{obs} - t_{obs}$	S_0/K					K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
$S_{obs} = 39$	**1/13**	0.0718	0.0396	0.0173	0.0073	0.0054	0.1127	0.0545	0.0229	0.0099	0.0068
$K_{obs} = 39$	**1/4**	0.0775	0.0445	0.0194	0.0070	0.0043	0.1104	0.0533	0.0247	0.0114	0.0072
	1/2	0.0744	0.0423	0.0178	0.0077	0.0049	0.1088	0.0597	0.0263	0.0107	0.0072
$S_{obs} = 39$	**1/13**	0.0642	0.0332	0.0144	0.0069	0.0053	0.1011	0.0440	0.0183	0.0128	0.0078
$K_{obs} = 36$	**1/4**	0.0741	0.0397	0.0179	0.0068	0.0051	0.0978	0.0426	0.0213	0.0134	0.0078
	1/2	0.0573	0.0321	0.0156	0.0070	0.0050	0.1042	0.0498	0.0207	0.0100	0.0072
$S_{obs} = 39$	**1/13**	0.0527	0.0181	0.0063	0.0069	0.0064	0.1140	0.0410	0.0162	0.0134	0.0081
$K_{obs} = 43$	**1/4**	0.0643	0.0307	0.0097	0.0044	0.0060	0.1189	0.0350	0.0220	0.0141	0.0079
	1/2	0.0387	0.0136	0.0103	0.0067	0.0051	0.1017	0.0419	0.0160	0.0114	0.0074

(b) Mean Absolute Percentage Error (MAPE)

Price Constraint		Squared Bias American call					Squared Bias American Put				
S_{obs}, K_{obs}	$T_{obs} - t_{obs}$	S_0/K					K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
$S_{obs} = 39$	**1/13**	0.00039	0.00122	0.00067	0.00001	0.00110	0.00000	0.00021	0.00085	0.00034	0.00278
$K_{obs} = 39$	**1/4**	0.00066	0.00207	0.00169	0.00005	0.00036	0.00001	0.00020	0.00117	0.00107	0.00304
	1/2	0.00040	0.00160	0.00074	0.00000	0.00046	0.00003	0.00044	0.00158	0.00081	0.00315
$S_{obs} = 39$	**1/13**	0.00021	0.00044	0.00026	0.00011	0.00111	0.00000	0.00000	0.00030	0.00170	0.00369
$K_{obs} = 36$	**1/4**	0.00049	0.00134	0.00089	0.00002	0.00090	0.00002	0.00001	0.00063	0.00237	0.00380
	1/2	0.00006	0.00041	0.00036	0.00000	0.00065	0.00000	0.00016	0.00049	0.00067	0.00311
$S_{obs} = 39$	**1/13**	0.00000	0.00000	0.00020	0.00100	0.00212	0.00011	0.00022	0.00004	0.00235	0.00395
$K_{obs} = 43$	**1/4**	0.00031	0.00061	0.00015	0.00046	0.00210	0.00020	0.00021	0.00073	0.00273	0.00380
	1/2	0.00012	0.00022	0.00015	0.00015	0.00087	0.00001	0.00000	0.00002	0.00105	0.00340

(c) Squared Bias

Price Constraint		$Variance/MSE$ American Call					$Variance/MSE$ American Put				
S_{obs}, K_{obs}	$T_{obs} - t_{obs}$	S_0/K					K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
$S_{obs} = 39$	**1/13**	0.7405	0.6583	0.8267	0.9964	0.6393	0.9879	0.8081	0.6745	0.8782	0.2642
$K_{obs} = 39$	**1/4**	0.6119	0.5310	0.6463	0.9787	0.8168	0.9601	0.8170	0.6164	0.7234	0.2870
	1/2	0.7482	0.5946	0.8129	0.9983	0.8258	0.8972	0.6725	0.5191	0.7508	0.2316
$S_{obs} = 39$	**1/13**	0.8236	0.8243	0.9066	0.9521	0.6371	0.9865	0.9968	0.8217	0.6360	0.2302
$K_{obs} = 36$	**1/4**	0.6847	0.6301	0.7823	0.9897	0.6789	0.9255	0.9838	0.7218	0.5235	0.2351
	1/2	0.9377	0.8322	0.8868	0.9998	0.7569	0.9887	0.8345	0.7684	0.7679	0.2617
$S_{obs} = 39$	**1/13**	0.9943	0.9988	0.6505	0.5670	0.4749	0.6921	0.6603	0.9692	0.5406	0.2532
$K_{obs} = 43$	**1/4**	0.7465	0.7135	0.8728	0.5575	0.3695	0.4320	0.5701	0.6989	0.5023	0.2498
	1/2	0.7298	0.5082	0.8924	0.9310	0.6925	0.9587	0.9971	0.9842	0.7238	0.2209

(d) Variance-MSE Ratio

Table 4
Pricing of American put and call options with time to maturity 0.5 and different levels of moneyness: Mean squared error (MSE), mean absolute percentage error (MAPE), squared bias and variance-MSE ratio of CONAA price estimates utilizing options with three different times to maturity (1/13, 1/2, 1/4) as constraint. MSE, MAPE and bias are determined relative to the Crank-Nicolson (750×750 grid) price for American put options and the Black-Scholes price for American call options, with the parameters $S_0 = 40$, $T - t = 0.5$, $r = 0.06$ and $\sigma = 0.2$, no dividends. The number of conducted price estimations is 1000. The variance is determined as difference between MSE and squared bias. Each cell presents the MSE/MAPE/squared bias/variance-MSE ratio for an option with a certain moneyness that is priced with CONAA, where the parameters S_{obs} and K_{obs} of the option price constraint for CONAA are given in the very left column. The observed market option prices are determined as Black-Scholes prices, with $T_{obs} - t_{obs}$ =1/13, 1/2, 1/4 (second column), $r = 0.06$, $\sigma = 0.2$. In CONAA, American call options are priced with a European call option price as constraint, American put options with a European put option price as constraint.

T	Method	MSE American Call					MSE American Put				
		S_0/K					K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
1/2	**CONAA**	0.0007	0.0008	0.0006	0.0024	0.0041	0.0003	0.0006	0.0015	0.0051	0.0051
	AC08	0.0018	0.0046	0.0054	0.0026	0.0019	0.0003	0.0014	0.0038	0.0037	0.0048
	HBS	0.0018	0.0044	0.0052	0.0017	0.0003	0.0005	0.0037	0.0312	0.1990	1.0259
	HCN	0.0018	0.0044	0.0052	0.0017	0.0003	0.0004	0.0019	0.0052	0.0025	0.0000
1/4	**CONAA**	0.0002	0.0004	0.0004	0.0014	0.0024	0.0001	0.0003	0.0007	0.0021	0.0014
	AC08	0.0002	0.0012	0.0023	0.0012	0.0015	0.0000	0.0004	0.0016	0.0016	0.0014
	HBS	0.0002	0.0015	0.0026	0.0005	0.0000	0.0000	0.0007	0.0096	0.0827	0.4094
	HCN	0.0002	0.0015	0.0026	0.0005	0.0000	0.0000	0.0005	0.0026	0.0006	0.0000
1/13	**CONAA**	0.0000	0.0001	0.0002	0.0009	0.0015	0.0000	0.0000	0.0002	0.0003	0.0002
	AC08	0.0000	0.0001	0.0006	0.0007	0.0012	0.0000	0.0000	0.0004	0.0003	0.0002
	HBS	0.0000	0.0001	0.0008	0.0000	0.0000	0.0000	0.0000	0.0016	0.0252	0.0487
	HCN	0.0000	0.0001	0.0008	0.0000	0.0000	0.0000	0.0000	0.0008	0.0000	0.0000

(a) Mean Squared Error (MSE)

T	Method	MAPE American Call					MAPE American Put				
		S_0/K					K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
1/2	**CONAA**	0.0492	0.0179	0.0065	0.0070	0.0063	0.1064	0.0406	0.0171	0.0134	0.0079
	AC08	0.0782	0.0456	0.0207	0.0074	0.0041	0.1162	0.0606	0.0283	0.0111	0.0077
	HBS	0.0810	0.0453	0.0209	0.0063	0.0019	0.1461	0.1091	0.0943	0.1028	0.1266
	HCN	0.0809	0.0453	0.0209	0.0063	0.0019	0.1224	0.0721	0.0332	0.0096	0.0000
1/4	**CONAA**	0.1201	0.0321	0.0082	0.0062	0.0053	0.2430	0.0637	0.0152	0.0088	0.0039
	AC08	0.1314	0.0554	0.0203	0.0056	0.0041	0.1982	0.0761	0.0239	0.0078	0.0041
	HBS	0.1310	0.0647	0.0226	0.0040	0.0006	0.1806	0.1089	0.0680	0.0699	0.0800
	HCN	0.1298	0.0646	0.0225	0.0040	0.0006	0.1684	0.0910	0.0315	0.0050	0.0000
1/13	**CONAA**	1.2260	0.1148	0.0102	0.0059	0.0048	1.6633	0.2073	0.0144	0.0035	0.0015
	AC08	1.1306	0.1212	0.0197	0.0052	0.0041	1.5265	0.1809	0.0205	0.0036	0.0014
	HBS	0.2952	0.1272	0.0240	0.0007	0.0000	0.3439	0.1680	0.0439	0.0397	0.0276
	HCN	0.2765	0.1246	0.0236	0.0007	0.0000	0.3234	0.1585	0.0289	0.0000	0.0000

(b) Mean Absolute Percentage Error (MAPE)

T	Method	Bias American Call					Bias American Put				
		S_0/K					K/S_0				
		0.8	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
1/2	**CONAA**	-0.0020	-0.0002	0.0148	0.0315	0.0460	0.0086	0.0137	-0.0076	-0.0481	-0.0618
	AC08	-0.0259	-0.0460	-0.0451	-0.0127	0.0127	-0.0058	-0.0209	-0.0428	-0.0338	-0.0604
	HBS	-0.0314	-0.0493	-0.0526	-0.0307	-0.0134	-0.0180	-0.0536	-0.1695	-0.4483	-1.0125
	HCN	-0.0313	-0.0493	-0.0525	-0.0306	-0.0134	-0.0138	-0.0317	-0.0524	-0.0366	0.0000
1/4	**CONAA**	0.0021	0.0034	0.0125	0.0220	0.0352	0.0040	0.0094	0.0039	-0.0215	-0.0287
	AC08	-0.0042	-0.0192	-0.0267	-0.0000	0.0232	0.0006	-0.0070	-0.0233	-0.0081	-0.0307
	HBS	-0.0100	-0.0287	-0.0388	-0.0170	-0.0038	-0.0043	-0.0224	-0.0923	-0.2865	-0.6398
	HCN	-0.0098	-0.0286	-0.0386	-0.0169	-0.0037	-0.0038	-0.0174	-0.0386	-0.0180	0.0000
1/13	**CONAA**	0.0000	0.0022	0.0078	0.0215	0.0324	-0.0000	0.0024	0.0070	0.0074	0.0030
	AC08	-0.0000	-0.0016	-0.0109	0.0165	0.0272	-0.0000	0.0003	-0.0084	0.0066	0.0005
	HBS	-0.0001	-0.0057	-0.0208	-0.0027	-0.0000	-0.0000	-0.0030	-0.0351	-0.1587	-0.2207
	HCN	-0.0001	-0.0055	-0.0204	-0.0026	-0.0000	-0.0000	-0.0027	-0.0208	0.0000	0.0000

(c) Bias

Table 5
Mean squared error (MSE), mean absolute percentage error (MAPE) and bias of CONAA, AC08, HBS and HCN price estimates for options with times to maturity $T - t = 1/2, 1/4, 1/13$ and moneyness 0.8, 0.9, 1.0, 1.1, 1.2 relative to the the Crank-Nicolson (750×750 grid) price for American put options and the Black-Scholes price for American call options, with the parameters $S_0 = 40$, $r = 0.06$ and $\sigma = 0.2$, no dividends. The number of conducted price estimations is 1000. In CONAA American call options are priced with a European call price as additional constraint, American put options with a European put option as constraint. The constraining market prices of the European options are Black-Scholes prices for the parameters: $S_{obs} = 39$, $K_{obs} = 39$, $T_{obs} - t_{obs} = T$, $r = 0.06$, and $\sigma = 0.2$. Historic volatility for HBS and HCN is estimated from the stock price paths. Each cell presents the MSE/MAPE/bias for an option with a certain moneyness and time to maturity that is priced with the method given in the second column.

		Squared Bias American Call					Squared Bias American Put				
		S_0/K					K/S_0				
T	Method	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
1/2	**CONAA**	0.0000	0.0000	0.0002	0.0010	0.0021	0.0001	0.0002	0.0001	023	0.0038
	AC08	0.0007	0.0021	0.0020	0.0002	0.0002	0.0000	0.0004	0.0018	0.0011	0.0036
	HBS	0.0010	0.0024	0.0028	0.0009	0.0002	0.0003	0.0029	0.0287	0.1970	1.0251
	HCN	0.0010	0.0024	0.0028	0.0009	0.0002	0.0002	0.0010	0.0027	0.0013	0.0000
1/4	**CONAA**	0.0000	0.0000	0.0002	0.0005	0.0012	0.0000	0.0001	0.0000	0.0005	0.0008
	AC08	0.0000	0.0004	0.0007	0.0000	0.0005	0.0000	0.0000	0.0005	0.0001	0.0009
	HBS	0.0001	0.0008	0.0015	0.0003	0.0000	0.0000	0.0005	0.0085	0.0821	0.4093
	HCN	0.0001	0.0008	0.0015	0.0003	0.0000	0.0000	0.0003	0.0015	0.0003	0.0000
1/13	**CONAA**	0.0000	0.0000	0.0001	0.0005	0.0011	0.0000	0.0000	0.0000	0.0001	0.0000
	AC08	0.0000	0.0000	0.0001	0.0003	0.0007	0.0000	0.0000	0.0001	0.0000	0.0000
	HBS	0.0000	0.0000	0.0004	0.0000	0.0000	0.0000	0.0000	0.0012	0.0252	0.0487
	HCN	0.0000	0.0000	0.0004	0.0000	0.0000	0.0000	0.0000	0.0004	0.0000	0.0000

(a) Squared Bias

		$Variance/MSE$ **American Call**					$Variance/MSE$ **American Put**				
		S_0/K					K/S_0				
T	Method	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
1/2	**CONAA**	0.9943	1.0000	0.6370	0.5936	0.4867	0.7598	0.7078	0.9614	0.5455	0.2501
	AC08	0.6174	0.5387	0.6246	0.9390	0.9164	0.9000	0.6832	0.5157	0.6873	0.2453
	HBS	0.4530	0.4540	0.4692	0.4610	0.4682	0.3280	0.2191	0.0786	0.0101	0.0008
	HCN	0.4545	0.4540	0.4704	0.4622	0.4695	0.4682	0.4610	0.4692	0.4525	NaN
1/4	**CONAA**	0.9746	0.9715	0.6360	0.6552	0.4892	0.7469	0.6974	0.9772	0.7772	0.3996
	AC08	0.9052	0.6811	0.6896	1.0000	0.6366	0.9916	0.8810	0.6574	0.9594	0.3451
	HBS	0.4201	0.4498	0.4244	0.4488	0.4187	0.3606	0.3201	0.1154	0.0081	0.0002
	HCN	0.4264	0.4511	0.4273	0.4528	0.4246	0.4187	0.4489	0.4244	0.4465	NaN
1/13	**CONAA**	0.9980	0.9105	0.6235	0.4827	0.3179	0.9997	0.7724	0.7759	0.8357	0.9616
	AC08	0.9993	0.9522	0.8007	0.6278	0.3850	0.9947	0.9951	0.8342	0.8703	0.9988
	HBS	0.4759	0.4191	0.4542	0.4175	0.4587	0.3880	0.3704	0.2265	0.0009	0.0000
	HCN	0.5577	0.4345	0.4649	0.4422	0.5212	0.4585	0.4176	0.4542	NaN	NaN

(b) Variance-MSE Ratio

Table 6

Squared bias and variance-MSE ratio of CONAA, AC08, HBS and HCN price estimates for options with times to maturity $T - t = 1/2, 1/4, 1/13$ and moneyness 0.8, 0.9, 1.0, 1.1, 1.2. Squared bias and MSE are determined relative to the the Crank-Nicolson (750×750 grid) price for American put options and the Black-Scholes price for American call options, with the parameters $S_0 = 40$, $r = 0.06$ and $\sigma = 0.2$, no dividends. Variance is determined as difference between MSE and squared bias. The number of conducted price estimations is 1000. In CONAA American call options are priced with a European call price as additional constraint, American put options with a European put option as constraint. The constraining market prices of the European options are Black-Scholes prices for the parameters: $S_{obs} = 39$, $K_{obs} = 39$, $T_{obs} - t_{obs} = T - t$, $r = 0.06$, and $\sigma = 0.2$. Historic volatility for H and HCN is estimated from the stock price paths. Each cell presents the squared bias/variance-MSE ratio for an option with a certain moneyness and time to maturity that is priced with the method given in the second column.

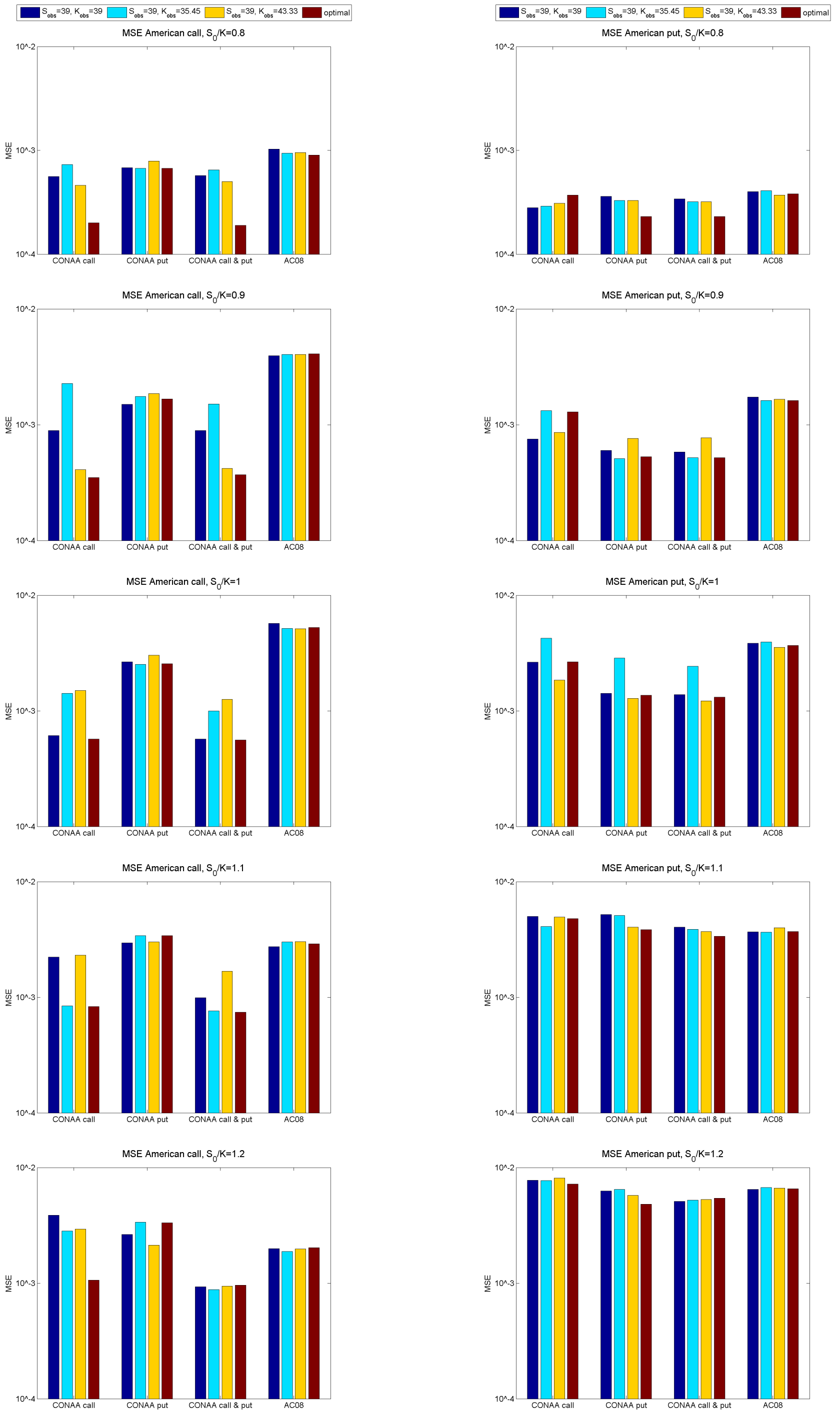

Fig. 1. Plots of the mean squared errors (MSE) in Table 1 of estimates of three versions of CONAA (with a call option as constraint - CONAA call, a put option as constraint - CONAA put, and both call and put option as constraints - CONAA c&p) and AC08. Each graph plots one column of Table 1, that is the results for one option with a certain moneyness (moneyness levels 0.8, 0.9, 1.0, 1.1, 1.2). The four bars for one method represent the MSEs of this method utilizing options with four different stock prices and strike prices as constraint: 1. $S_{obs} = 39$, $K_{obs} = 39$; 2. $S_{obs} = 39$, $K_{obs} = 35.45$; 3. $S_{obs} = 39$, $K_{obs} = 43.33$; 4. optimal, that is $S_{obs} = S_0 = 40$, $K_{obs} = K$ where K is the strike price of the priced option (AC08 does not include these additional option price constraints). Market option values are calculated with Crank-Nicolson (American put) and Black-Scholes (American calls). Constraining option prices are calculated with the Black-Scholes formulas for European call and put options, parameters: $T - t = T_{obs} - t_{obs} = 0.5$, $r = 0.06$, $\sigma = 0.2$, no dividends. The scale is logarithmic.

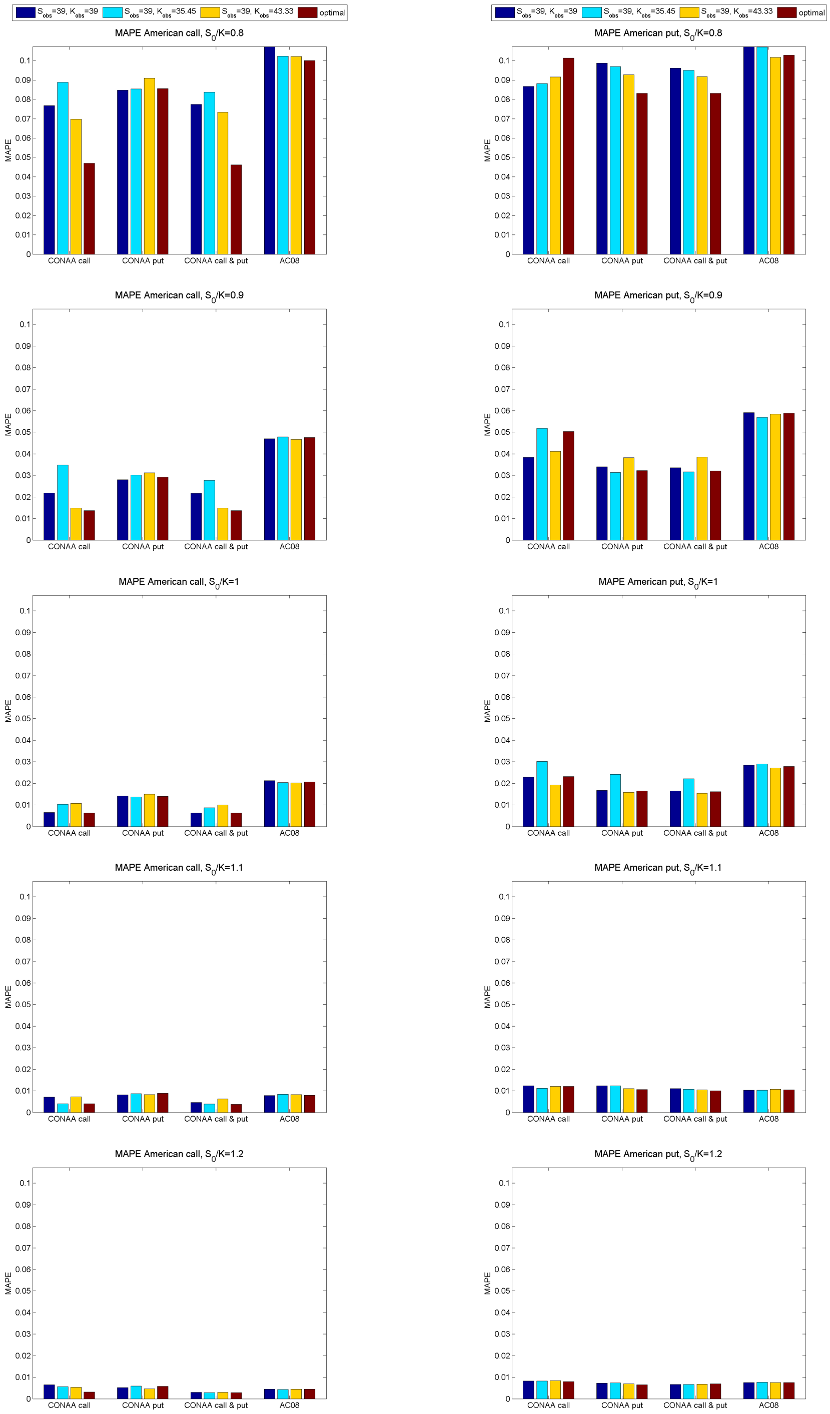

Fig. 2. Plots of the mean absolute percentage errors (MAPE) in Table 1 of estimates of three versions of CONAA (with a call option as constraint - CONAA call, a put option as constraint - CONAA put, and both call and put option as constraints - CONAA c&p) and AC08. Each graph plots one column of Table 1, that is the results for one option with a certain moneyness (moneyness levels 0.8, 0.9, 1.0, 1.1, 1.2). The four bars for one method represent the MAPEs of this method utilizing options with four different stock prices and strike prices as constraint: 1. $S_{obs} = 39$, $K_{obs} = 39$; 2. $S_{obs} = 39$, $K_{obs} = 35.45$; 3. $S_{obs} = 39$, $K_{obs} = 43.33$; 4. optimal, that is $S_{obs} = S_0 = 40$, $K_{obs} = K$ where K is the strike price of the priced option (AC08 does not include these additional option price constraints). Market option values are calculated with Crank-Nicolson (American put) and Black-Scholes (American calls). Constraining option prices are calculated with the Black-Scholes formulas for European call and put options, parameters: $T - t = T_{obs} - t_{obs} = 0.5$, $r = 0.06$, $\sigma = 0.2$, no dividends.

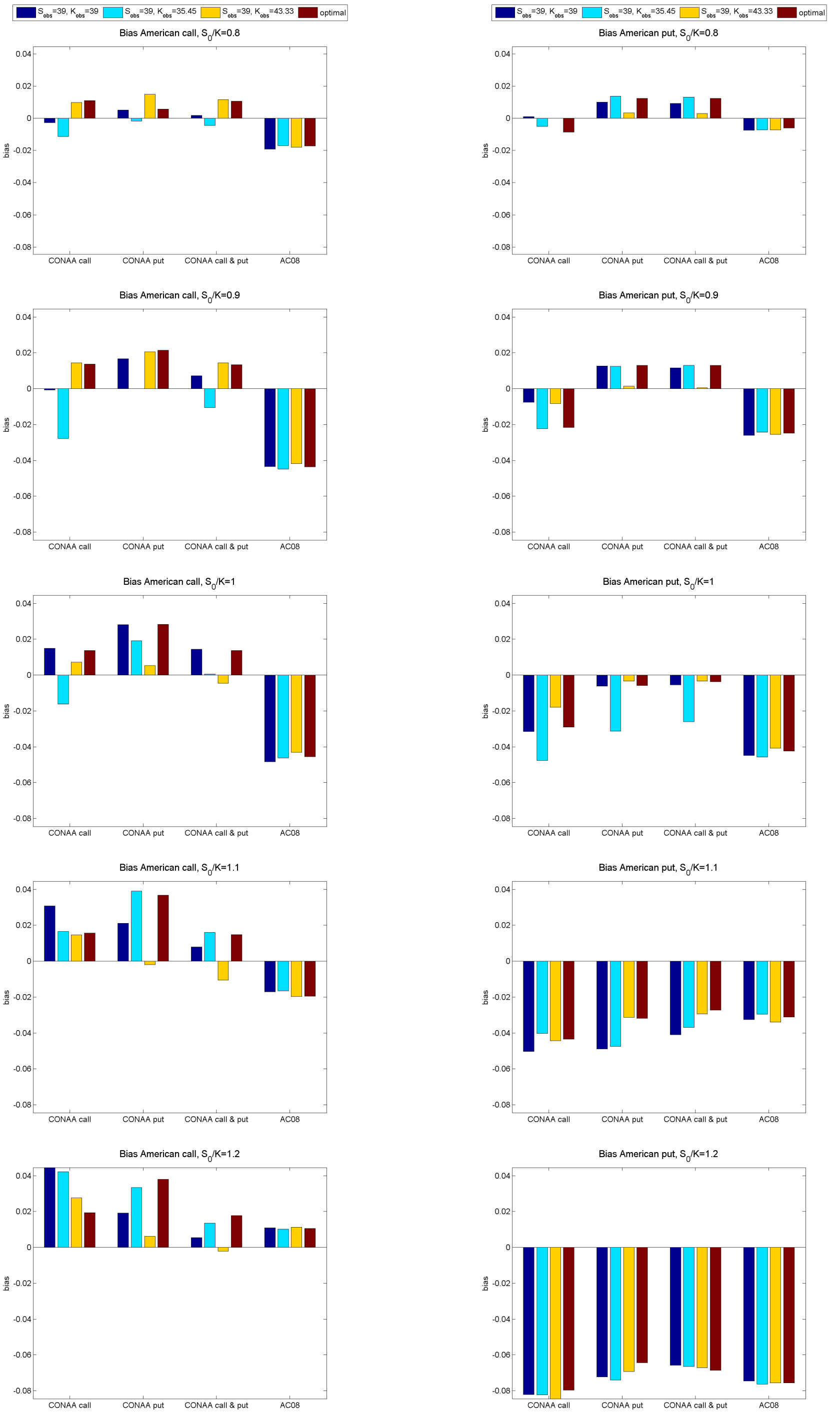

Fig. 3. Plots of the bias values in Table 2 of estimates of three versions of CONAA (with a call option as constraint - CONAA call, a put option as constraint - CONAA put, and both call and put option as constraints - CONAA c&p) and AC08. Each graph plots one column of Table 2, that is the results for one option with a certain moneyness (moneyness levels 0.8, 0.9, 1.0, 1.1, 1.2). The four bars for one method represent the bias values of this method utilizing options with four different stock prices and strike prices as constraint: 1. $S_{obs} = 39$, $K_{obs} = 39$; 2. $S_{obs} = 39$, $K_{obs} = 35.45$; 3. $S_{obs} = 39$, $K_{obs} = 43.33$; 4. optimal, that is $S_{obs} = S_0 = 40$, $K_{obs} = K$ where K is the strike price of the priced option (AC08 does not include these additional option price constraints). Market option values are calculated with Crank-Nicolson (American put) and Black-Scholes (American calls). Constraining option prices are calculated with the Black-Scholes formulas for European call and put options, parameters: $T - t = T_{obs} - t_{obs} = 0.5$, $r = 0.06$, $\sigma = 0.2$, no dividends.

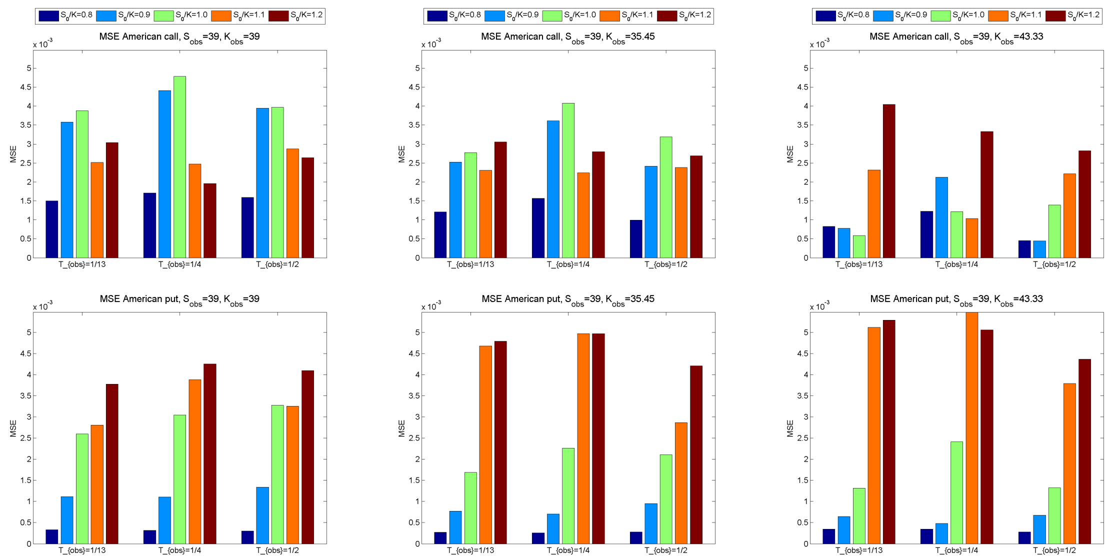

Fig. 4. Plots of the mean squared errors (MSE) in Table 4 (a) of CONAA estimates determined with option price constraints with time to maturity $T_{obs} - t_{obs} = 1/13, 1/4, 1/2$. Each plot shows the MSEs of the three maturities over the levels of moneyness 0.8, 0.9, 1.0, 1.1, 1.2 given the option price constraint with S_{obs} and K_{obs} (1. $S_{obs} = 39$, $K_{obs} = 39$; 2. $S_{obs} = 39$, $K_{obs} = 36$; 3. $S_{obs} = 39$, $K_{obs} = 43$). Market option prices are calculated with Crank-Nicolson (American put) and Black-Scholes (American call). Constraining option prices are calculated with the Black-Scholes formulas for European call and put options. Parameters: $T - t = T_{obs} - t_{obs} = 0.5$, $r = 0.06$, $\sigma = 0.2$, no dividends. American call options are priced with a European call option as price constraint, American put options with a European put option as price constraint.

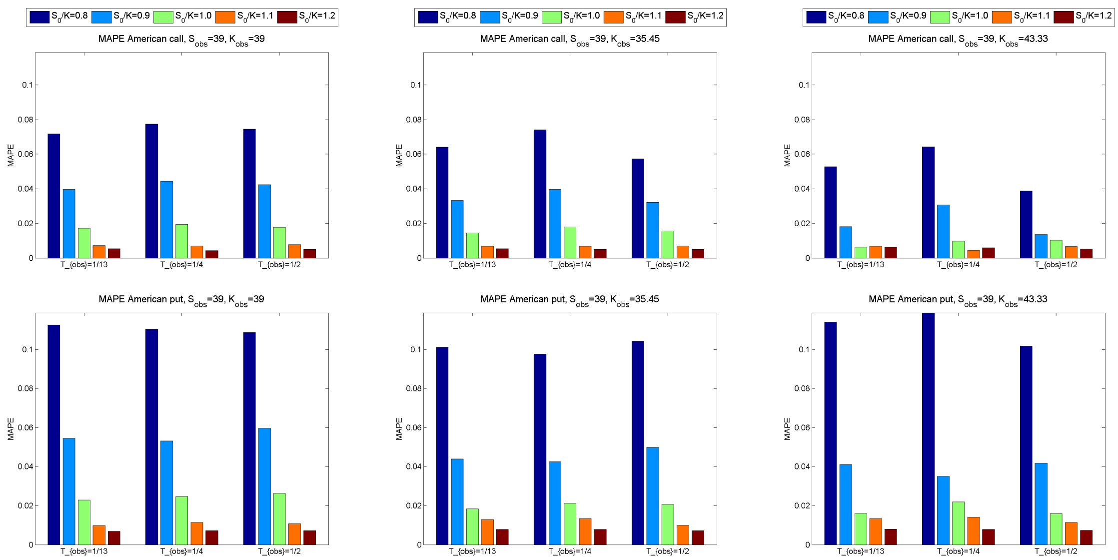

Fig. 5. Plots of the mean absolute percentage errors (MAPE) in Table 4 (b) of CONAA estimates determined with option price constraints with time to maturity $T_{obs} - t_{obs} = 1/13, 1/4, 1/2$. Each plot shows the MSEs of the three maturities over the levels of moneyness 0.8, 0.9, 1.0, 1.1, 1.2 given the option price constraint with S_{obs} and K_{obs} (1. $S_{obs} = 39$, $K_{obs} = 39$; 2. $S_{obs} = 39$, $K_{obs} = 36$; 3. $S_{obs} = 39$, $K_{obs} = 43$). Market option prices are calculated with Crank-Nicolson (American put) and Black-Scholes (American call). Constraining option prices are calculated with the Black-Scholes formulas for European call and put options. Parameters: $T - t = T_{obs} - t_{obs} = 0.5$, $r = 0.06$, $\sigma = 0.2$, no dividends. American call options are priced with a European call option as price constraint, American put options with a European put option as price constraint.

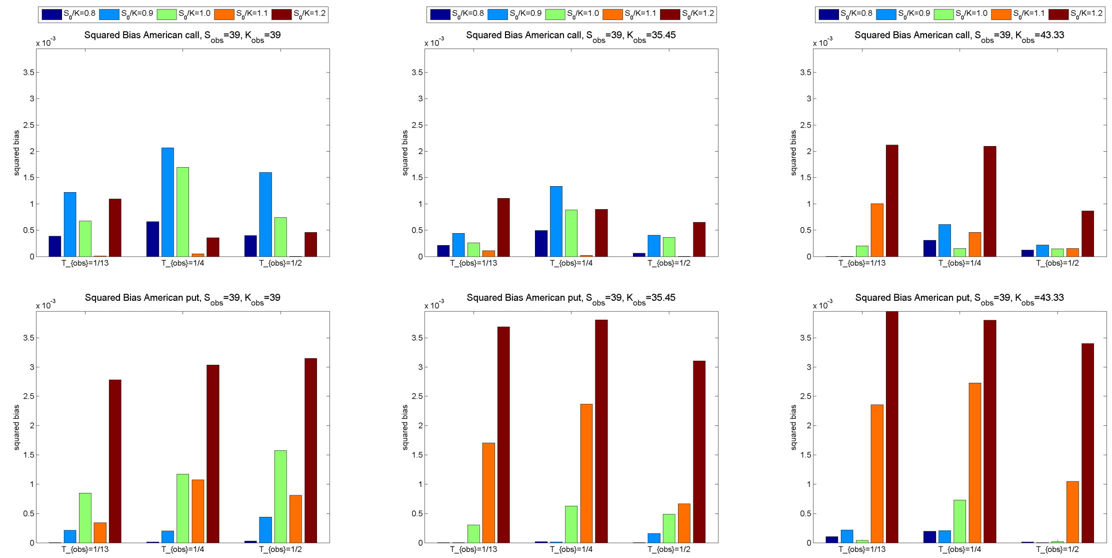

Fig. 6. Plots of the squared bias in Table 4 (c) of CONAA estimates determined with option price constraints with time to maturity $T_{obs} - t_{obs} = 1/13, 1/4, 1/2$. Each plot shows the MSEs of the three maturities over the levels of moneyness 0.8, 0.9, 1.0, 1.1, 1.2 given the option price constraint with S_{obs} and K_{obs} (1. $S_{obs} = 39$, $K_{obs} = 39$; 2. $S_{obs} = 39$, $K_{obs} = 36$; 3. $S_{obs} = 39$, $K_{obs} = 43$). Market option prices are calculated with Crank-Nicolson (American put) and Black-Scholes (American call). Constraining option prices are calculated with the Black-Scholes formulas for European call and put options. Parameters: $T - t = T_{obs} - t_{obs} = 0.5$, $r = 0.06$, $\sigma = 0.2$, no dividends. American call options are priced with a European call option as price constraint, American put options with a European put option as price constraint.

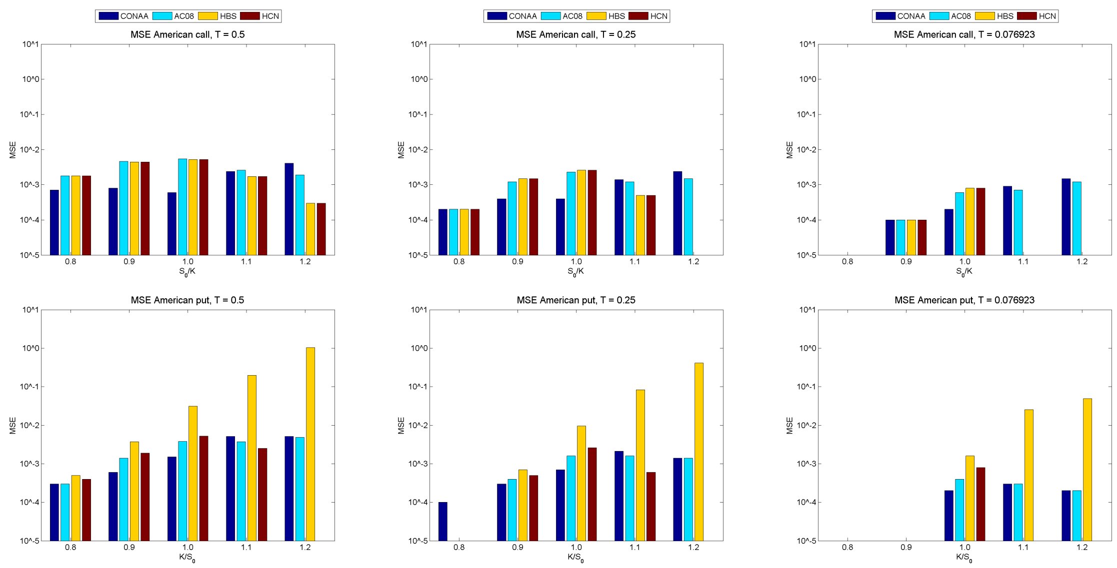

Fig. 7. Plots of the mean squared errors (MSE) in Table 5 (a) of CONAA, AC08, HBS and HCN price estimates for options with times to maturity $T - t = 1/2, 1/4, 1/13$ and moneyness 0.8, 0.9, 1.0, 1.1, 1.2. One plot refers to one of the six big cells of Table 5 (a) and shows the MSEs for one time to maturity of the American call (put) option over all levels of moneyness. For each moneyness the result of each of the four methods is represented by a bar. Market option prices are calculated with Crank-Nicolson (American put) and Black-Scholes (American call) with the parameters $S_0 = 40$, $r = 0.06$, $\sigma = 0.2$, no dividends. Constraining option prices are calculated with Black-Scholes for European call and put options with the parameters $T_{obs} - t_{obs} = T - t$, $S_{obs} = 39$, $K_{obs} = 39$ $r = 0.06$, $\sigma = 0.2$, no dividends. American call options are priced with a European call option as price constraint, American put options with a European put option as price constraint.

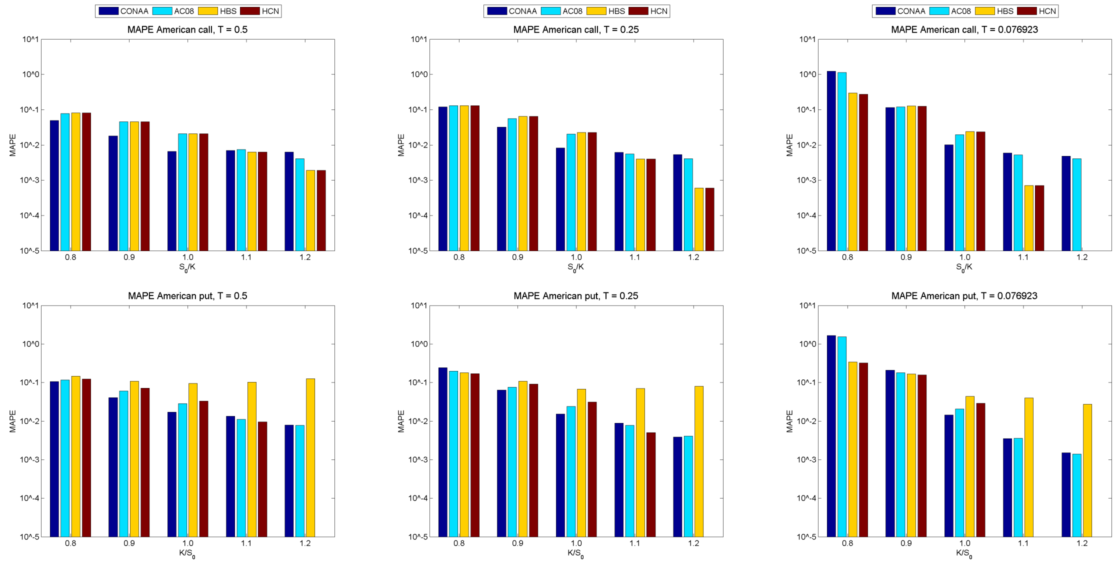

Fig. 8. Plots of the mean absolute percentage errors (MAPE) in Table 5 (b) of CONAA, AC08, HBS and HCN price estimates for options with times to maturity $T - t = 1/2, 1/4, 1/13$ and moneyness 0.8, 0.9, 1.0, 1.1, 1.2. One plot refers to one of the six big cells of Table 5 (b) and shows the MAPEs for one time to maturity of the American call (put) option over all levels of moneyness. For each moneyness the result of each of the four methods is represented by a bar. Market option prices are calculated with Crank-Nicolson (American put) and Black-Scholes (American call) with the parameters $S_0 = 40$, $r = 0.06$, $\sigma = 0.2$, no dividends. Constraining option prices are calculated with Black-Scholes for European call and put options with the parameters $T_{obs} - t_{obs} = T - t$, $S_{obs} = 39$, $K_{obs} = 39$ $r = 0.06$, $\sigma = 0.2$, no dividends. American call options are priced with a European call option as price constraint, American put options with a European put option as price constraint. The scale is logarithmic.

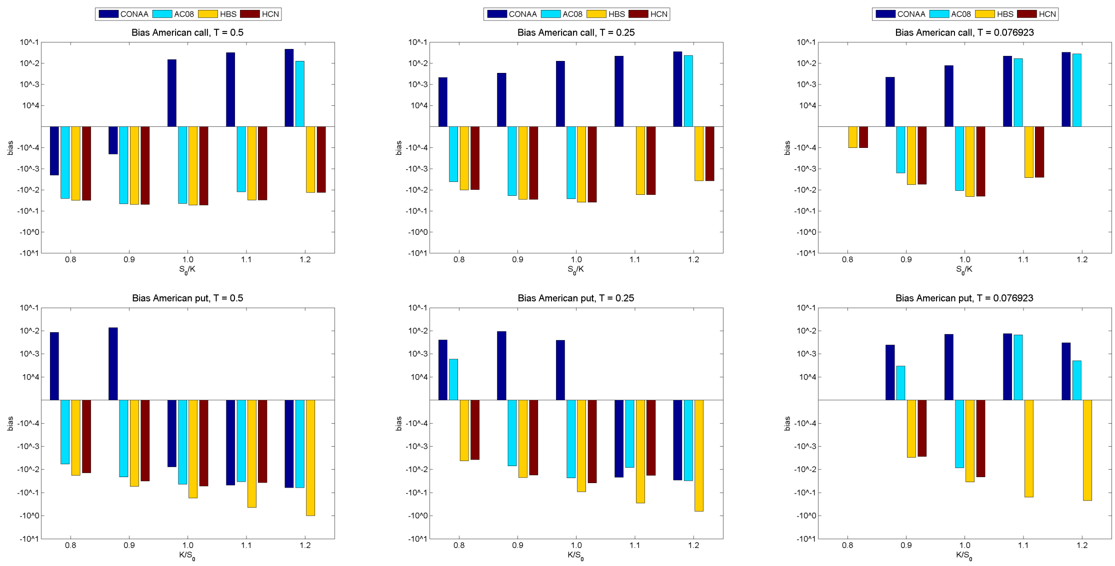

Fig. 9. Plots of the bias in Table 5 (c) of CONAA, AC08, HBS and HCN price estimates for options with times to maturity $T - t = 1/2, 1/4, 1/13$ and moneyness 0.8, 0.9, 1.0, 1.1, 1.2. One plot refers to one of the six big cells of Table 5 (c) and shows the bias values for one time to maturity of the American call (put) option over all levels of moneyness. For each moneyness the result of each of the four methods is represented by a bar. Market option prices are calculated with Crank-Nicolson (American put) and Black-Scholes (American call) with the parameters $S_0 = 40$, $r = 0.06$, $\sigma = 0.2$, no dividends. Constraining option prices are calculated with Black-Scholes for European call and put options with the parameters $T_{obs} - t_{obs} = T - t$, $S_{obs} = 39$, $K_{obs} = 39$ $r = 0.06$, $\sigma = 0.2$, no dividends. American call options are priced with a European call option as price constraint, American put options with a European put option as price constraint. The scale is logarithmic.

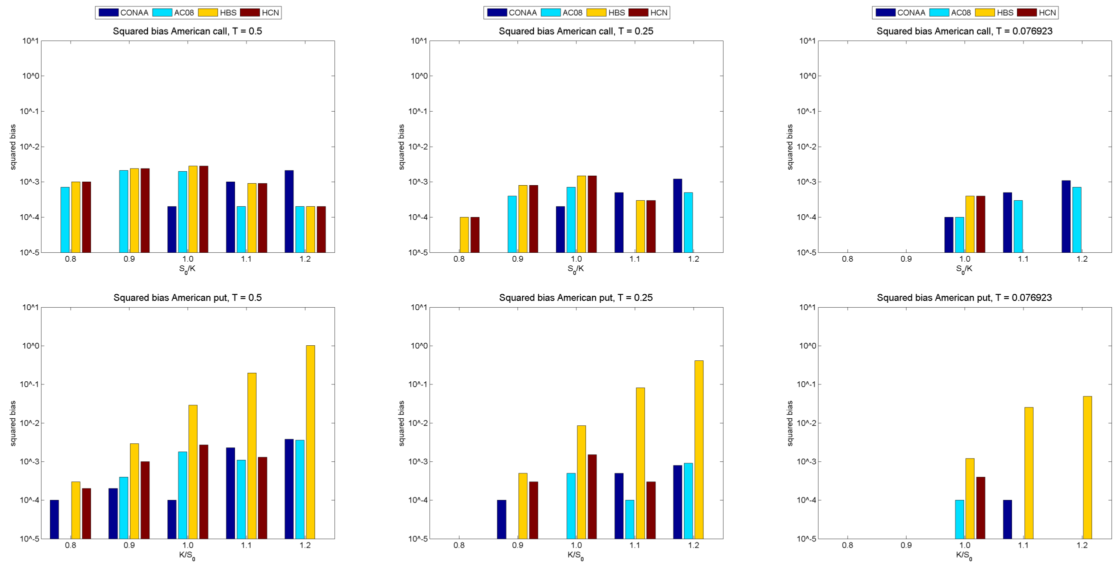

Fig. 10. Plots of the squared bias in Table 5 (d) of CONAA, AC08, HBS and HCN price estimates for options with times to maturity $T - t = 1/2, 1/4, 1/13$ and moneyness 0.8, 0.9, 1.0, 1.1, 1.2. One plot refers to one of the six big cells of Table 5 (d) and shows the squared bias values for one time to maturity of the American call (put) option over all levels of moneyness. For each moneyness the result of each of the four methods is represented by a bar. Market option prices are calculated with Crank-Nicolson (American put) and Black-Scholes (American call) with the parameters $S_0 = 40$, $r = 0.06$, $\sigma = 0.2$, no dividends. Constraining option prices are calculated with Black-Scholes for European call and put options with the parameters $T_{obs} - t_{obs} = T - t$, $S_{obs} = 39$, $K_{obs} = 39$ $r = 0.06$, $\sigma = 0.2$, no dividends. American call options are priced with a European call option as price constraint, American put options with a European put option as price constraint. The scale is logarithmic.