SMOOTH SIMULTANEOUS CALIBRATION OF THE LMM TO CAPLETS AND COTERMINAL SWAPTIONS

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ABSTRACT. We introduce a new calibration methodology that allows perfect fitting of the displaced diffusion LIBOR market model to caplets and co-terminal swaptions, whilst avoiding global optimizations. The approach works by regarding a forward rate as a difference of swap-rates and then bootstrapping through rates one by one.

1. Introduction

Whilst the LIBOR market model is easy to calibrate to caplet prices, there are many such calibrations, and the simple naive approach using flat volatilities has often been criticized. This is both because of failure to calibrate other instruments and lack of realism.

What properties should a calibration methodology have?

- Ability to fit a chosen subset of swaptions and caplets.
- Time homogeneity: the future implied volatilities should look as similar to today's as possible.
- Correlation: the instantaneous correlations between rates should be as similar to historical as possible.
- Stability: small changes in input parameters should not result in large changes in the calibration.

Here we present a methodology which allows calibration to caplets and co-terminal swaptions in a smooth way whilst exogenously specifying the correlation between rates across each time step. The calibration is distinguished by being as time-homogeneous as possible whilst fitting the caplets and co-terminal swaptions. We work with the displaced diffusion LIBOR market model throughout.

We draw on some ideas of Gallucio *et al* [8]. In that paper, they show how to approximately calibrate a full-factor co-terminal swap-rate market model to caplets and swaptions simultaneously. Whilst they say that the extension to reduced factor models is straightforward, they do not actually specify how to do it; calibrating a full factor model and then factor reducing will not be sufficient as it is not clear how to do this whilst preserving both caplet and co-terminal swaption prices. The key to their method is the observation that a forward rate can be written approximately as the weighted sum of

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two neighbouring co-terminal swap rates, and this leads to an approximation formula for caplet volatilities in terms of the swaption volatilities and their correlation.

They suggest starting with a time-homogeneous fit to swaption volatilities, and work through the caplets one by one. Swap-rate volatilities are perturbed to match the caplet implied volatility in such a way as to preserve time-homogeneity as much as is possible within the class of perturbations up to the reset time of the previous swap-rate but *not* across the final step.

Our approach is more geometric and we work principally with the much more popular displaced diffusion LIBOR market model. Rather than working with a parametrized form to obtain time homogeneity, we instead find the piecewise constant time-dependent volatility functions which are closest to time homogeneous amongst all choices that price the caplets and swaptions correctly. This is done by using a combination of simple geometric arguments and a series of stable one-dimensional optimizations.

In particular,

- we use the equivalence between the co-terminal swap-rate market model and the LIBOR market model, as in [20], to apply the methodology to the both models;
- we work with displaced log-normal dynamics allowing the calibration of a model with skew;
- we show how by using piece-wise constant volatilities the method can be made to apply directly to low-factor LIBOR market models;
- we adjust the method to maximize time homogeneity across all time steps for each rate and solve using geometric arguments;
- we show how the approximation formula for the caplet implied volatility can be improved;
- we eliminate all approximation bias for the caplet by using an iterative procedure.

A precise methodology for low-factor models is important since the computational complexity of the LIBOR market model and the co-terminal swap-rate market model are proportional to the number of factors, [17, 19, 25].

The calibration of the LIBOR market is a problem of great interest to practitioners, and there therefore has been many papers on the topic. A good overview is given [3]. Many of these rely on high-dimensional optimizations which we avoid here. One popular method due to Pedersen [24] uses piecewise constant volatilities to calibrate to a selection of caplets and swaptions. Wu has shown how Lagrange multipliers can be used to fit [31]. Brigo and Mercurio use a cascade algorithm to fit the entire swaption matrix by successively solving quadratics, but this tends to suffer from imaginary volatilities [6]. Brace and Womersley have shown how to use semi-definite programming [5]. The main issue with their method is that the computational complexity is too great to be effective when the number of rates is large. Rebonato also discusses various methodologies in [27].

As an application of our results, we study the factor dependence of Bermudan swaptions. In particular, we study 10 non-call 1 (10NC1) Bermudan swaptions with varying strikes in both the LIBOR market model and swap-rate market model, for varying numbers of factors. We develop both lower and upper bounds. We see that with our calibration methodology that there is very little factor dependence and that the LIBOR market model and swap-rate market model give very similar prices when holding the prices of at-the-money caplets and swaptions fixed.

The factor dependence of Bermudan swaptions is a problem with a controversial history: Longstaff, Santa Clara and Schwartz, [21], argued that the use of low-factor models results in too low prices and therefore banks were effectively throwing away money. Anderson and Andreasen, [1], argued that they were mistaken as they were using the wrong sort of model and not calibrating correctly. It is the question of calibration that is at the heart of the issue: dependence on factors is only meaningful in terms of what else is held constant. Anderson and Andreasen did a best fit to the entire swaption matrix and found little dependence.

Svenstrup [30] and Choy, Dun and Schlögl [7] ran similar tests and also found little dependence. Here we fix the at-the-money caplet and co-terminal swaptions implied volatilities whilst varying the factors. We also examine the difference between prices in swap-rate market models and the LIBOR market model. We find little variation between the models and little dependence on the number of factors. This is similar to the results in [20] where the two market models gave similar prices with a slight dependence on factors but the caplet prices were not held fixed.

2. Defining a calibration

Let us fix notation. We have times $0 = t_{-1} < t_0 < t_1 < \cdots < t_j < \cdots < t_n$. Forward rate f_j runs from time t_j to t_{j+1}. Swap-rate SR_j runs from t_j to t_n with the underlying times being $t_j, \ldots, t_n$. In this article, we are interested in simulating the joint distribution of the rates at the times t_l.

In the F-factor displaced diffusion market model with rates g_j, we take

$$\frac{d(g_j + a_j)}{g_j + a_j} = \mu_j dt + \sum_{k=1}^{F} b_{jk}(t) dW_k, \tag{2.1}$$

with the terms μ_j determined by no arbitrage conditions, and the weightings b_{jk} are constant across each time step, that is they are constant on each interval (t_l, t_{l+1}); this constancy within time steps is important to ensure that the effective covariance of the (displaced) log rates (ignoring drifts) across each step is of order F. Whilst one can take the displacements a_j to be rate specific, we shall take them to be the same for all rates.

The objective of a calibration is therefore to choose the terms b_{jk} for each time-step. Following [20], we will map calibrations between the LIBOR market model and the co-terminal swap-rate model by using the log-Jacobian, Z, of the forward-rate to swap-rate map.

In particular, we have that the swap-rate SR_j is a function of the forward rates f_l for $l \geq j$, and we can consider the mapping

$$\Theta : (f_1, \ldots, f_n) \mapsto (\mathrm{SR}_1, \ldots, \mathrm{SR}_n),$$

this mapping is well-known to be invertible (see e.g. [19] for an algorithm) and has upper triangular Jacobian since rate SR_j does not depend on f_l for $l < j$. For a formula for the entries of the Jacobian see [13]. The inverse of an upper triangular matrix is also upper triangular, so the inverse mapping also has upper triangular Jacobian.

We can write, ignoring drifts,

$$d \log(\mathrm{SR}_j + a) = \sum_l \frac{f_l + a}{\mathrm{SR}_j + a} \frac{\partial \mathrm{SR}_j}{\partial f_l} d \log(f_l + a). \tag{2.2}$$

In vector terms, we can write

$$d \log(\mathrm{SR} + a) = Z d \log(f + a), \tag{2.3}$$

with Z the Jacobian of the mapping of displaced log-rates, with individual terms as in (2.2).

Whilst Z will be state-dependent, we can use its value with the initial forward rates, $Z(0)$, as an approximation. This is the essence of the Hull–White approximation, [10], and was applied in [20] to show the equivalence of forward-rate and co-terminal swap-rate market models.

In practical terms, if we are given a pseudo-square root,

$$B_i = (b^i_{jk}), \tag{2.4}$$

for each time step for the displaced log-forwards then we can use

$$E_i = Z(0) B_i \tag{2.5}$$

as a pseudo-square root for the displaced log-swap-rates for that step. Applying $Z(0)^{-1}$ will equally give us a mapping in the opposite direction. Note that B_i and E_i have the same number of columns so this mapping preserves the number of factors. It will occasionally be useful to work with the covariance matrices of the displaced log rates, we set

$$C_i^f = B_i B_i^t, \tag{2.6}$$

$$\tilde{C}_i^f = \sum_{j \leq i} C_j^f, \tag{2.7}$$

$$C_i^{\mathrm{SR}} = E_i E_i^t, \tag{2.8}$$

$$\tilde{C}_i^{\mathrm{SR}} = \sum_{j \leq i} C_j^{\mathrm{SR}}. \tag{2.9}$$

Our objective in this paper is to choose a B_i with F columns for each time-step in such a way that caplets are priced correctly, and the implied swaption volatilities given by using $Z(0)B_i$ also match the market values. Thus our calibration will be accurate up to the accuracy of the Hull–White approximation. In fact, our approach will be to calibrate the co-terminal swap-rate market model and only shift to working with the forward-rate model at the end.

Let $\hat{\sigma}_f(i)$ be the (displaced) implied volatility of the caplet on forward rate i, and similar for $\hat{\sigma}_{\mathrm{SR}}(i)$. Clearly, we must have

$$\hat{\sigma}_f(n-1) = \hat{\sigma}_{\mathrm{SR}}(n-1).$$

Thus our conditions are

$$\tilde{C}^{f}_{i,ii} = \hat{\sigma}_f(i)^2 t_i, \tag{2.10}$$

$$\tilde{C}^{\mathrm{SR}}_{i,ii} = \hat{\sigma}_{\mathrm{SR}}(i)^2 t_i, \tag{2.11}$$

for each i.

3. Chopping Hull and White

We will first calibrate in swaption terms and then deform to fit to caplets. Our approximation for the variance of rate $\log(f_j + a)$ is from the previous section

$$\sum_{k,l \geq j} Z(0)^{-1}_{jk} \tilde{C}^{\mathrm{SR}}_{j,kl} Z(0)^{-1}_{jl}.$$

Since $Z(0)^{-1}$ is upper triangular, the sum is not over all k, l.

The essence of the approach of Gallucio *et al* is that f_j is approximately a linear combination of SR_j and SR_{j+1} and they therefore drop the terms $Z(0)^{(-1)}_{jl}$ with $l > j+1$. This is justified by the fact that the dropped terms are roughly one thousand times smaller than the retained ones. They call this truncated Hull–White and show that it is effective.

In fact, the size of the discarded terms depends on the slope of the yield curve. If all forward rates are the same then they are zero, cf [13] where correction terms disappear for flat curves. For an upwards sloping curve, they will be small and negative. Note that the number of terms dropped in the approximation for SR_j is $(n-j)^2 - 4$, so even if they are one thousand times smaller, their total impact can be significant when they all have the same sign.

We will therefore work with a different approximation which we shall call *chopped Hull–White.* We let

$$w_{j,0} = Z(0)^{-1}_{jj} + \sum_{k=j+2}^{n} Z(0)^{-1}_{jk}, \tag{3.1}$$

$$w_{j,1} = Z(0)^{-1}_{j,j+1}. \tag{3.2}$$

Our approximation for the variance of $\log(f_j + a)$ is then

$$\tilde{\sigma}_j^2 t_j = w_{j,0}^2 \tilde{C}^{\mathrm{SR}}_{j,jj} + 2 w_{j,0} w_{j,1} \tilde{C}^{\mathrm{SR}}_{j,j,j+1} + w_{j,1}^2 \tilde{C}^{\mathrm{SR}}_{j,j+1,j+1}. \tag{3.3}$$

4. Splitting the problem

We have reduced the problem to choosing a pseudo-square root for each time step. A pseudo-square root has n rows and F columns. We will need a pseudo-square root for each of the n steps. This means that our calibration procedure will need to determine $n^2 F$ elements (whilst some of these will be zero because of reset times, the order is correct). We are calibrating to $2n-1$ instruments. Clearly, there is a great deal of freedom here.

We will therefore split each pseudo-square root into a volatility component and a correlation component. So we write

$$E^i_{j,k} = \sigma^j_i \rho^i_{j,k}, \tag{4.1}$$

with

$$\sum_{k=1}^{F} (\rho^i_{j,k})^2 = 1, \tag{4.2}$$

that is $\rho^i_{j,k}$ is the pseudo-square root of an F-factor correlation matrix. Note that σ^j_i will really be the standard deviation rather than the volatility as it already includes the square-root of time.

We shall regard the correlation pseudo-root matrix as exogenously given. We discuss ways of making the choice in Section 6. We are left with determining the terms σ^j_i.

5. Using time homogeneity

In section 4, we saw that by exogenously specifying the correlation matrix, the calibration problem is reduced to determining piecewise constant swap-rate volatilities. However, we still have $n(n+1)/2$ quantities to use to match $2n-1$ observations. One way of imposing additional structure is to require time homogeneity. The essential idea is that one should expect the shape of observed implied swaption volatilities as a function of maturity to be roughly constant over time.

It is important to realize that the swap-rates in the co-terminal model are intrinsically different from each other since each has a different tenor; this is different from the LIBOR market model where rates only differ by maturity.

We therefore assume that we are given not just the implied volatilities of the swap-rates underlying the model, but also the implied volatilities of swaptions with the same tenors at each of the reset dates t_i.

For each swap-rate SR_j, we fit a function $\sigma_j(t_j - t)$, to the implied volatilities of the swaptions with tenors $t_n - t_j$ and maturities t_i with $i \le j$. Gallucio *et al* suggest using the functional form

$$\bar{\sigma}_j(s) = (a + bs)e^{-cs} + d,$$

and obtain through a least-squares fit a choice of a, b, c and d for each j. Since we are most interested in the implied volatility of SR_j we multiply the entire function by a constant to match its volatility exactly. For discussion of this functional form and its primitive see [27].

In this paper, we want a reduced-factor calibration so in order to avoid any decorrelation arising from time-dependence of rates within steps, we replace $\bar{\sigma}_j(s)$ by a piecewise constant function taking the value σ_l^j in the interval (t_l, t_{l+1}) which has the same square integral across that interval. Note that if we allowed decorrelation to arise from time dependence, then the number of factors could increase, and we would no longer have an F-factor calibration.

The objective of the remainder of our calibration will be to deform σ_l^j in such a way that the total variance is preserved whilst matching the implied caplet volatilities. Our objective will be to minimize the magnitude of this deformation in a least-squares sense.

6. Correlation

There are a number of ways to specify the correlation matrix. For the methodology in this paper, the way it is obtained is not important, since we exogenously specify it and then scale using volatility. The important aspect of its specification is that our approach requires an F-factor correlation matrix for the logs of the forward rates across each step. Note that when we discuss correlation, strictly speaking we are addressing the correlation of the evolution of the logs without drifts.

This means that we must specify a pseudo-root matrix of size n by F for each step. Note that we are specifying the correlations across a finite time interval rather than the instantaneous correlations. This fact in combination with the use of piece-wise constant volatilities ensures that the model truly has F-factors across each step.

One approach due to Rebonato, [27], is to take the instantaneous correlation between rates i and j with reset times t_i and t_j to be

$$L + (1 - L)e^{-\beta||t_i - t|^\gamma - |t_j - t|^\gamma|},$$

where L, β and γ are calibrated to historical data. The correlation matrix for a step is then found by integrating this and normalizing by the implied standard deviations. This will generally yield a full-factor correlation matrix. We will use this approach with the further simplification that we use the correlations at the middle of the step rather than carrying out an integration.

We can factor reduce, however. To do so, we compute the eigenvalues, λ_i, and eigenvectors, ϕ_i. We reorder to obtain

$$\lambda_i \geq \lambda_{i+1}.$$

We then set the last $n - F$ eigenvalues to zero. We also set any remaining negative eigenvalues to zero (this only occurs if the original matrix is not positive definite.) We let A be the matrix with column j equal to $\sqrt{\lambda_j}\phi_j$.

The matrix

$$C = AA^t,$$

will not be a correlation matrix since its diagonal elements need not be one, however, we can scale the random variables by dividing by their standard deviations to get a correlation matrix. This is equivalent to dividing each row of A by its size as a vector.

This yields an F-factor correlation matrix for forward rates. An $F-$ factor correlation matrix for co-terminal swap-rates can be found via conjugation with Z, and again scaling to make standard deviations equal to 1.

7. The solving methodology

Following Gallucio *et al*, we calibrate to the caplet on rate f_j, by modifying the time-dependence of the volatility of rate SR_{j+1}. They suggest introducing two multipliers q_j and r_j the first q_j multiplying the volatility of SR_{j+1} up to time t_j and the second across the time interval (t_j, t_{j+1}). Since f_j sets at time t_j, r_j does not affect its implied volatility. One can therefore choose q_j to match the volatility of f_j and then r_j to match the volatility of SR_{j+1}. At each stage, they assume the volatility of rate SR_l for $l \leq j$ has already been determined. The volatility of SR_0 is not affected by calibration to caplets.

Our approach is more subtle and rather than scaling the volatility of SR_{j+1}, we consider the set of all piece-wise constant volatility functions, and find the element that matches caplet and swaption prices and is closest to the time homogeneous solution. We start with the vectors σ_l^j and replace them successively with new vectors $\tilde{\sigma}_l^j$.

Suppose therefore that we have determined the volatility vectors of $\mathrm{SR}_0, \dots, \mathrm{SR}_{j-1}$ and have already calibrated to the swaptions on them, and the caplets on $f_0, \dots, f_{j-2}$. We will now determine the volatity vector, σ_l^j, of SR_j and calibrate to the swaption which SR_j underlies and the caplet on f_{j-1}.

If $\tilde{C}_j^{\mathrm{SR}}$ is the total (displaced log-rate) swap-rate covariance matrix out to time t_j implied by the volatility functions $sig\tilde{m}a_l^k$ for $k < j$ and σ_l^k for $k \geq j$, then using the chopped Hull–White approximation, the implied variance of $\log(f_{j-1} + a)$ is

$$C^f_{j-1,jj} = w^2_{j-1,0}\tilde{C}^{\mathrm{SR}}_{j-1,j-1,j-1} + 2w_{j-1,0}w_{j-1,1}\tilde{C}^{\mathrm{SR}}_{j-1,j-1,j} + w^2_{j-1,1}\tilde{C}^{\mathrm{SR}}_{j-1,j,j}.$$

Since at this stage, we are only varying the value of the vector σ_l^j, the variance is a quadratic in our variables.

We can look at our problem in a more abstract fashion. We have a vector σ_l^j with $j + 1$ elements. We wish to solve two quadratics: the swaption price and the caplet price. The solution set should therefore be an $j-1$-dimensional submanifold. We want the point on that sub-manifold which is closest to the undeformed time-homogeneous solution.

First, we need to obtain a better understanding of the geometry of the problem. We work with a vector

$$z = (z_0, z_1, \ldots, z_j) \in \mathbb{R}^{j+1},$$

instead of σ_l^j for notational simplicity. The constraint that the swaption is priced correctly becomes

$$\sum_{i \leq j} z_i^2 = R^2,$$

where R is the standard deviation of the log swap-rate. This clearly describes a sphere of radius R.

Using the chopped Hull-White approximation, the constraint that the caplet is priced correctly becomes

$$A + \sum_{i<j} B_i z_i + \sum_{i<j} z_i^2 = S^2,$$

where S is the standard deviation of the log forward rate underlying the caplet divided by the coefficient of the quadratic term, for some A, B_i. We can rewrite this as

$$A - \frac{1}{4} \sum_{i<j} B_i^2 + \sum_{i<j} \left(z_i + \frac{B_i}{2} \right)^2 = S^2. \tag{7.1}$$

Considering the first j coordinates alone, this describes a sphere centred at $-B/2$. In the last coordinate there is no constraint, so we have a cylinder.

Our problem is therefore to find the point on the intersection of the sphere for the swaption and the cylinder for the caplet, that lies closest to a given point x representing the time-homogeneous solution.

To simplify the problem we can exploit symmetries. Let O denote a rotation of $\mathbb{R}^{j+1}$ that fixes the origin and $-B/2$. We also assume that O does not change the final coordinate. So we can regard O as isometry of $\mathbb{R}^j$ that acts trivially in the last coordinate. We have that O preserves both the cylinder and the sphere.

This means that having found a point in the intersection we can rotate via mappings such as O to get a point close to x. Or turning the problem round we can rotate to make x have a simple value, solve and then rotate back.

To solve we therefore transform the problem to make B have a simple form. Thus we rotate the first j coordinates so that

$$B = \lambda e_j,$$

i.e. is zero in all coordinates except j.

We then carry out a second rotation in the first $j-1$ coordinates (which fixes B) to get $x_i = 0$ for $i < j-1$, we also keep $x_{j-2} \geq 0$. The best solution now clearly has $z_i = 0$ for $i < j-1$, and so we have reduced to a three dimensional:

$$z_{j-1}^2 + z_j^2 + z_{j+1}^2 = R^2, \tag{7.2}$$

$$z_{j-1}^2 + (z_j - \lambda)^2 = S^2, \tag{7.3}$$

and we have to minimize

$$(z_{j-1} - x_{j-1})^2 + (z_j - x_j)^2 + (z_{j+1} - x_{j+1})^2.$$

We can use z_j to parametrize the solution set of the first two equations. We simply put

$$z_{j-1} = \sqrt{S^2 - (z_j - \lambda)^2}, \tag{7.4}$$

$$z_{j+1} = \sqrt{R^2 - z_{j-1}^2 - z_j^2}. \tag{7.5}$$

We will perform a 1d minimization using z_j to get the z closest to x in rotated coordinates.

To do the minimization, it is necessary to determine what values of z_j are acceptable. For z_{j-1} to exist, clearly we must have

$$-S \leq z_j - \lambda \leq S. \tag{7.6}$$

We also must have that z_{j+1} is real. Using the formula for z_{j-1}, we have

$$z_{j-1}^2 = S^2 - (z_j^2 - \lambda)^2 = S^2 - z_j^2 + 2\lambda z_j - \lambda^2.$$

So

$$z_{j+1}^2 = R^2 - z_j^2 - z_{j-1}^2 = R^2 - S^2 + \lambda^2 - 2\lambda z_j.$$

Since $\lambda > 0$, the reality of z_j is equivalent to

$$z_j \leq \frac{R^2 - S^2 + \lambda^2}{2\lambda}. \tag{7.7}$$

If

$$\frac{R^2 - S^2 + \lambda^2}{2\lambda} \geq S + \lambda,$$

then this is a non-condition from (7.6). This will occur if and only if

$$R \geq S + \lambda,$$

which corresponds to the case that the intersection of the cylinder with $z_{j+1} = 0$ lies inside the sphere.

Having found the bounds for z_j, we optimize using it as a parameter, and compute z_{j-1} and z_{j+1}. To carry out the optimization, we used a golden section search (see e.g. [26],) on the interval

$$\left[S - \lambda, \min\left(S + \lambda, \frac{R^2 - S^2 + \lambda^2}{2\lambda}\right)\right].$$

As this is a one-dimensional finding of the closest point on the intersection of the sphere and the cylinder, it will be stable. After carrying out the optimization, we then undo our rotation to find the solution in the original coordinates.

Clearly, to carry out this program, we need explicit formulas for S, A and B. We can obtain these from equation (3.3) by dividing through by the coefficient of the quadratic part.

The coefficient of the quadratic part is $\theta = w_{j-1,1}^2$ before we divide. So S equals the (implied) standard deviation of the log forward divided by $\sqrt{\theta}$. For A we get

$$w_{j-1,0}^2 \tilde{C}_{j-1,j-1}^{\mathrm{SR}}/\theta,$$

and

$$B_i = 2w_{j-1,0}w_{j,1}\tilde{\sigma}_i^{j-1}\rho_{j,j-1}^i/\theta.$$

8. Rotation construction

The next question is how to construct the rotation. A rotation is up to a reflection an orthogonal matrix. An orthogonal matrix is a change in choice of orthonormal basis, so we need to find an orthonormal basis, h_i, such that h_j is a multiple of B, and h_{j-1} is a multiple of B and x', where x' is x projected onto the first j coordinates. In fact, an orthogonal matrix is enough here but we can always reflect in an unimportant coordinate to get a rotation.

To find the orthonormal basis, we first construct a basis g_i as follows.

- Let $g_1 = B$,
- If x' is linearly independent to B we set g_2 to equal x'.
- We now keep adding e_i if it is linearly independent until we have a basis.
- We perform Gram-Schmidt on the ordered basis g.
- The new basis has h_1 being a multiple of B and x' being in the span of h_1 and h_2. We can relabel so this will hold for h_{j-1} and h_j instead.

Our orthogonal matrix now has rows equal to h_j. This will map B to a multiple of e_1. (We can relabel to get e_j.) With z_j mapping to a multiple of e_1 and e_2 as needed. We refer the reader who is unfamiliar with Gram–Schmidt to [29].

9. Robustification

Whilst in our experience, the method presented here does not fail with real market data, and in the event of a failure we would advocate checking the data's consistency, nevertheless for reassurance and for the ability to specify inputs, one wishes a method that would never fail.

There are two ways in which the method can fail:

- the cylinder can be empty, this corresponds to its radius squared being negative;
- the cylinder and the sphere may not intersect.

In the first case, using the found volatility of SR_i there is no specification of time-dependent volatility for SR_{i+1} that will price the caplet on f_i correctly. The closest matching will then be obtained by taking a cylinder of zero radius, that is a line. We therefore simply reset the radius-squared to zero when it is negative.

For the second case, we can match the caplet volatility on f_i or match the swaption volatility on SR_{i+1} but not both. In this case, we have to choose which to prioritize. Taking a point on the sphere will get the swaption correct, and a point on the cylinder for the caplet. We therefore take the point, α, on the sphere closest to the cylinder, and β on the cylinder closest to the sphere. We then pick a parameter θ expressing our priority and use

$$z = (1-\theta)\alpha + \theta\beta.$$

Note that if we use $\theta = 0.5$, we will assign equal priorities.

If these solutions are adopted then the algorithm will always find a reasonable fit even when inputs are bad, but it will not necessarily be perfect. As long as no errors of the first sort occur then it is possible to fit either caplets or swaptions perfectly.

10. Final stage

Once we have carried out the procedure of the previous sections, we have matched the volatilities of the co-terminal swaptions in the co-terminal swap-rate market model precisely, and we have matched the caplet prices in that model up to the accuracy of the chopped Hull–White approximation.

We use the untruncated $Z(0)^{-1}$ to map our co-terminal swap calibration into a LIBOR market model calibration. This will price the co-terminal swaptions correctly up to the accuracy of the full Hull–White approximation. For caplets, we have a slight mismatch since they have been calibrated using the chopped approximation but have been mapped using the full approximation. This typically results in an error of implied volatility of less than 0.1%.

We now discuss how to remove that error. Our solution is to modify the inputs to the calibration algorithm. We can regard the calibration algorithm as a map from a set of vector of caplet volatilities and swaption volatilities to a vector of forward-rate pseudo-roots. We then compose this map with the map to implied caplet and Hull–White approximated swaption volatilities. Call this map Φ.

A perfect calibration (up to the accuracy of the Hull–White approximation) would be a fixed point of Φ. Our construction has given a point

$$(\sigma_{f_0}, \sigma_{f_1}, \dots, \sigma_{f_{n-1}}, \sigma_{\mathrm{SR}_0}, \sigma_{\mathrm{SR}_1}, \dots, \sigma_{\mathrm{SR}_{n-1}})$$

where Φ fixes the last n points but the first n are perturbed. So

$$\Phi((\sigma_{f_0}, \sigma_{f_1}, \dots, \sigma_{f_{n-1}}, \sigma_{\mathrm{SR}_0}, \sigma_{\mathrm{SR}_1}, \dots, \sigma_{\mathrm{SR}_{n-1}}) = (\nu_{f_0}, \nu_{f_1}, \dots, \nu_{f_{n-1}}, \sigma_{\mathrm{SR}_0}, \sigma_{\mathrm{SR}_1}, \dots, \sigma_{\mathrm{SR}_{n-1}}). \quad (10.1)$$

We therefore restart the calibration procedure with the vector of caplet volatilities given by

$$\sigma_{f,j}.\frac{\sigma_{f,j}}{\nu_{f,j}}.$$

Whilst in theory one could this as many times as necessary, the accuracy after one extra calibration is typically of the order $1e-6$ in implied volatility, and this is more than sufficiently accurate.

Note that one could play a similar trick with the swaption implied volatilities if one wished to use a more accurate approximation than Hull–White or even a full pricing simulation.

11. Tenor incompatibility

In some markets, the frequency of payments in the caplet market and the swaption market are different. For example, in the Euro market caplets are on six-month rates but the fixed leg of a swap is yearly. Our objective in this section is therefore to extend our methodology to allow the calibration of a LIBOR market model to the case where the swap fixed leg payment is some multiple of the caplet period.

In order to solve an inverse problem such as calibration, it is often the best to start with the forward problem: how to quickly get the prices of swaptions with different periods in a LIBOR market model. We will do this and then show how the scaling trick can be used to carry out the calibration.

To get the prices of swaptions with the same periods, we can apply the Hull-White approximation. Our approach is therefore to go via a LIBOR market model with the longer period. Once we have approximated its dynamics the approximate prices of the swaptions will be immediate.

How to get the dynamic of LIBOR rates with a long period from those with a short period is detailed in Section 6.16 of [6] where explicit formulas can be found. The approach is via coefficient freezing. Suppose rate f_i has rates $f_{i,1}$ and $f_{i,2}$ underlying it with accruals τ_1 and τ_2. We then have

$$\frac{1}{1+f_i(\tau_1+\tau_2)} = \frac{1}{1+f_{i,1}\tau_1}\frac{1}{1+f_{i,2}\tau_2}.$$

Rearranging, we can therefore write

$$f_i = g_i(f_{i,1}, f_{i,2}),$$

with g a smooth function. We can approximate the volatility part of the dynamics of $\log(f_i + a)$ via

$$d(\log f_i + a) = \frac{\partial \log(g_i + a)}{\partial \log(f_{i,1} + a)} d\log(f_i + a),$$

this is enough to deduce the pseudo-square of the covariance matrix of the logs across each time step. We thus have a calibration of the model with larger steps and applying Hull–White we can obtain the prices of the long period swaptions.

4.37%
4.39%
4.43%
4.48%
4.55%
4.66%
4.74%
4.86%
4.92%

Table 1. Initial values of one-year forward rates

14.6%
14.3%
14.0%
13.6%
13.2%
12.8%
12.5%
12.1%
11.8%

Table 2. Caplet ATM volatilities on one-year forward rates

To calibrate, we use the scaling trick again, similarly to Section 10, and proceed as follows:

- Calibrate the short period LIBOR market model to caplets and co-terminal short period swaptions using market long period swaption volatilities as if they were the correct input swaption volatilities. (Interpolate to get any swaptions where the volatility is not known.)
- Map this calibration into the long period equivalent, and calculate model implied long period swaption volatilities.
- Scale the input swaption volatilities by the ratio between the market long period swaption volatilities and the model implied ones.
- Repeat until the desired level of accuracy is achieved.

12. An example calibration

We use some recent Euro data (from late 2007) to price a 10NC1 payer Bermudan swaption. Since fixed rate payments are annual in the Euro market, we have 9 underlying forward, or co-terminal swap-, rates. The times for our simulations are just

$$t_j = j + 1 \quad \text{for} \quad j = 0, \ldots, 9.$$

We work with zero displacement for simplicity and transparency. Our forward rates are as in Table 1. The at-the-money (ATM) caplets volatilities

	1	2	3	4	5	6	7	8	9
1	14.6%	14.6%	14.5%	14.3%	14.2%	13.8%	13.5%	13.2%	13.0%
2	14.3%	14.1%	13.9%	13.8%	13.6%	13.3%	13.0%	12.8%	12.6%
3	14.0%	13.7%	13.5%	13.3%	13.1%	12.8%	12.6%	12.4%	12.3%
4	13.6%	13.4%	13.1%	12.9%	12.7%	12.5%	12.3%	12.1%	12.0%
5	13.2%	13.0%	12.7%	12.5%	12.3%	12.1%	12.0%	11.8%	11.7%
6	12.9%	12.6%	12.4%	12.2%	12.0%	11.9%	11.7%	11.6%	11.5%
7	12.5%	12.2%	12.0%	11.8%	11.7%	11.6%	11.4%	11.3%	11.3%
8	12.1%	11.9%	11.7%	11.5%	11.4%	11.4%	11.2%	11.1%	11.1%
9	11.8%	11.6%	11.4%	11.3%	11.2%	11.1%	11.0%	11.0%	11.0%

TABLE 3. Swaption ATM volatilities. Tenor is by column and maturity by row.

a	0.057	0.064	0.067	0.067	0.066	0.060	0.055	0.050	0.048
b	0.047	0.027	0.012	-0.0032	7.44E-06	4.412E-05	8.31E-06	-5.41E-06	2.51E-06
c	0.500	0.427	0.356	0.213	0.306	0.296	0.278	0.259	0.236
d	0.084	0.084	0.085	0.087	0.089	0.089	0.089	0.088	0.088

TABLE 4. Fit values of a, b, c and d.

L	0.5
β	0.2
γ	0.5

TABLE 5. Parameters used for the correlation structure.

	1	2	3	4	5	6	7	8	9
1	0.02123	0.02129	0.02132	0.02134	0.02108	0.01996	0.01860	0.01768	0.01690
2	0.02066	0.01921	0.01811	0.01753	0.01696	0.01637	0.01557	0.01508	-
3	0.01789	0.01662	0.01550	0.01483	0.01421	0.01393	0.01346	-	-
4	0.01496	0.01425	0.01346	0.01290	0.01235	0.01224	-	-	-
5	0.01257	0.01233	0.01189	0.01150	0.01106	-	-	-	-
6	0.01081	0.01087	0.01071	0.01047	-	-	-	-	-
7	0.00956	0.00978	0.00981	-	-	-	-	-	-
8	0.00871	0.00900	-	-	-	-	-	-	-
9	0.00813	-	-	-	-	-	-	-	-

TABLE 6. Variances for each time step for each swap-rate

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on them are in Table 2 and we display the ATM swaption volatilities in Table 3.

We fit an abcd curve to the swaption implied volatilities in such a way as to get the co-terminal swaption volatilities precisely correct. The values are displayed in Table 4. These are then turned into piecewise constant volatilities which imply variances per step that are displayed in Table 6.

To go further requires a choice of correlation structure. We work with an L, β, γ structure as discussed in Section 6, with parameters as in Table 5.

	1	2	3	4	5	6	7	8	9
1	0.02101	0.01991	0.02119	0.02010	0.02115	0.01941	0.01875	0.01673	0.01690
2	0.02044	0.01941	0.01758	0.01779	0.01625	0.01728	0.01432	0.01604	-
3	0.01793	0.01611	0.01593	0.01380	0.01487	0.01187	0.01456	-	-
4	0.01457	0.01479	0.01266	0.01395	0.01126	0.01395	-	-	-
5	0.01261	0.01132	0.01255	0.01010	0.01212	-	-	-	-
6	0.01019	0.01180	0.00918	0.01282	-	-	-	-	-
7	0.00997	0.00760	0.01172	-	-	-	-	-	-
8	0.00745	0.01241	-	-	-	-	-	-	-
9	0.01036	-	-	-	-	-	-	-	-

TABLE 7. Variances for each time step for each swap-rate after full-factor calibration

.

factors	RMS deformation
1	1.06%
2	1.01%
3	0.95%
4	0.89%
9	0.79%

TABLE 8. Root-mean-square deformation to swaption volatilities for varying numbers of factors

LS training paths	65536,
Lower bound estimation paths	2^{19},
Inner paths for upper bound	1024,
Outer paths for upper bound	1024.

TABLE 9. Numbers of paths used for simulations.

For the full-factor model, we display the piecewise constant variances after the fitting procedure in Table 7.

We look at the cases with numbers of factors $1, 2, 3, 4$ and 9. In all these cases, the swaption volatilities were correct to floating point accuracy and the caplets were correct with a root-mean-square error of less than $3E-5$ that is 0.3 basis points. (This is in terms of the Hull–White approximation when doing the caplets in the SMM and the swaptions in the LMM.) For the reduced factor models, we present the root-mean-square change in volatility per step in Table 8. Since the total variance is preserved for each swap-rate each deformation up on a step will imply that another one goes down. The deformation decreases with the number of factors suggesting that a realistic correlation structure helps compatibility between caplet and swaption markets. However, even in the worst case the root-mean-square deformation is only just over 1% whilst achieving perfect fits to caplets and swaptions.

factors	SMM lower	LMM lower	SMM upper	LMM upper
1	0.0500	0.0500	0.0503	0.0507
2	0.0501	0.0501	0.0504	0.0508
3	0.0501	0.0502	0.0505	0.0508
4	0.0501	0.0501	0.0505	0.0509
9	0.0501	0.0501	0.0504	0.0507

Table 10. Upper and lower bounds for Bermudan swaption with varying numbers of factors in the LIBOR market model and swap-rate market model. Strike is 4%.

Factors	SMM lower	LMM lower	SMM upper	LMM upper
1	0.0200	0.0200	0.0206	0.0208
2	0.0201	0.0201	0.0208	0.0210
3	0.0202	0.0202	0.0207	0.0210
4	0.0202	0.0202	0.0209	0.0210
9	0.0202	0.0202	0.0208	0.0209

Table 11. Upper and lower bounds for Bermudan swaption with varying numbers of factors in the LIBOR market model and swap-rate market model. Strike is 5%.

factors	SMM lower	LMM lower	SMM upper	LMM upper
1	0.0082	0.0082	0.0086	0.0087
2	0.0083	0.0083	0.0087	0.0088
3	0.0083	0.0083	0.0087	0.0089
4	0.0083	0.0083	0.0088	0.0087
9	0.0083	0.0083	0.0087	0.0088

Table 12. Upper and lower bounds for Bermudan swaption with varying numbers of factors in the LIBOR market model and swap-rate market model. Strike is 6%.

factors	SMM lower	LMM lower	SMM upper	LMM upper
1	0.0035	0.0035	0.0036	0.0036
2	0.0035	0.0035	0.0037	0.0037
3	0.0035	0.0035	0.0036	0.0036
4	0.0035	0.0035	0.0037	0.0037
9	0.0035	0.0035	0.0037	0.0037

Table 13. Upper and lower bounds for Bermudan swaption with varying numbers of factors in the LIBOR market model and swap-rate market model. Strike is 7%.

13. Bermudan swaptions

In this section, we price 10NC1 payer Bermudan swaptions using the calibration from the previous section. Our objective is to assess the dependence on factors and whether model choice matters. We price in both the LIBOR market model and the swap-rate market model. Our lower bounds are produced using the least-squares method of Longstaff and Schwartz [22]. For basis functions, we use three European swaptions: the option on the next swap-rate, the option on the final caplet and a swaption in the middle for the swap-rate market model. For the LIBOR market model, the value of the swaptions is not easily available so we use the first three powers of the next swap-rate, and its annuity.

For upper bounds we use the Anderson–Broadie extension [2] of Rogers' method [28] which involves sub-simulations. We also apply the refinement from [18] that allows the non-running of sub-simulations when out-of-the-money.

We present the parameters used in Table 9. We use the Mersenne random number generator for the generation of the training paths and a Sobol generator using the Joe–Kuo D7 initialization numbers [15] for the pricing (along with the usual low-discrepancy refinements, see eg [11].) We are using many more paths than typically used to ensure full convergence and that the lower bounds are converged to within 0.1 basis points.

For the upper bounds, we use the Mersenne twister with different seeds for both inner and outer simulations. The use of the two-pass approach in the lower bound and the uncorrelated sub-simulations in the upper bounds means that we have unbiased estimates of upper and lower bounds. The upper bounds have standard errors that always less than 1 basis point and typically around 0.5 basis points.

We present results for a payers Bermudan swaption with strikes 4%, 5%, 6% and 7% in Tables 10, 11, 12 and 13. In all cases, the upper and lower bounds are close together reassuring us that the lower bounds are highly accurate. For each strike, there is virtually no variation in the lower bound prices with factors and model-type.

14. Conclusion

We have presented a robust method of calibrating the displaced- diffusion LIBOR and swap-rate market models that allows simultaneous fitting of caplets and co-terminal swaptions. It works by construction in a reduced-factor displaced model and only requires one-dimensional convex optimizations. With this calibration methodology, there is little factor dependence in the prices of Bermudan swaptions.

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