

An Analysis of the True Notional Bond System Applied to the CBOT T-Bond Futures*

Ramzi Ben-Abdallah[†] Hatem Ben-Ameur
HEC Montréal, Canada CREF and GERAD, HEC Montréal, Canada

Michèle Breton
CREF and GERAD, HEC Montréal, Canada

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Abstract

The main purpose of this paper is to apply the True Notional Bond System (TNBS) proposed by Oviedo (2006) for the theoretical pricing of the Chicago Board of Trade Treasury-bond futures, one of the most traded derivatives in the world. This system is proposed as an alternative to the current conversion factor system (CFS), whose imperfections are well known. In this paper, we price the CBOT T-bond futures as well as all its embedded delivery options under the CFS and the TNBS in a stochastic interest rate framework. Our pricing procedure is an adaptation of the Dynamic Programming algorithm described in Ben-Abdallah et al. (2007), giving the value of the futures contract under the TNBS as a function of time and current short-term interest rate. Numerical illustrations, provided under the Vasicek and CIR models, compare the value of the options embedded in the CBOT T-bond futures under the two systems.

JEL Classification: C61; C63; G12; G13.

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[†]HEC Montréal, 3000 chemin de la Côte Sainte-Catherine, Montreal, Canada, H3T 2A7; Tel: (514) 340-6000#2469; Fax: (514) 340-5634; E-mail: ramzi.ben-abdallah@hec.ca.

1 Introduction

A futures contract is an agreement between two investors traded on an exchange to sell or to buy an underlying asset at some given time in the future, called the delivery date, for a given price, called the futures price. By convention, at the time the futures is written (the inception date), the futures price is known and sets the value for both parties to zero. A futures contract is marked to market once a day to eliminate counterparty risk. Thus, at the end of each trading day, the futures contract is rewritten at a new settlement price, that is, the closing futures price, and the difference with the last settlement futures price is subtracted (resp. added) from the short (resp. long) trader account.

The Treasury Bond futures traded on the Chicago Board of Trade (the CBOT T-bond futures in the sequel) is the most actively traded and widely used futures contract in the United States, largely because of its ability to hedge long-term interest rate risk. It calls for the delivery of \$100,000 of a long-term governmental bond. The notional or reference bond is a bond with a 6% coupon rate and a maturity of 20 years. Delivery months (DM) are March, June, September and December.

Since the notional bond is a hypothetical bond that is generally not traded in the market place, the short has the option to choose which bond to deliver among a deliverable set fixed by the CBOT. The actual delivery day within the delivery month is also at the option of the short. These two delivery privileges offered to the short trader are known as the quality option (or choosing option) and the timing option.

The quality option allows the delivery of any governmental bond with at least 15 years to maturity or earliest call. To make the delivery fair for both parties, the price received by the short trader is adjusted according to the quality of the T-bond delivered. This adjustment is made via a set of conversion factors defined by the CBOT as the prices of the eligible T-bonds at the first delivery date under the assumption that its yield to maturity is 6%. The T-bond actually delivered by the short trader is called the cheapest-to-deliver (CTD). The timing option allows the short trader to deliver early within a delivery month according to special features, that is, the delivery sequence and the end-of-month delivery rule. The

delivery sequence consists of three consecutive business days: The position day, the notice day, and the delivery day. During the position day, the short trader can declare his intention to deliver until up to 8:00 p.m., while the CBOT closes at 2:00 p.m. (Central Standard Time). On the notice day, the short trader has until 5:00 p.m. to state which T-bond will be actually delivered. The delivery then takes place before 10:00 a.m. of the delivery day, against a payment based on the settlement price of the position day (adjusted according to the conversion factor). Finally, during the last seven business days before maturity, trading on the T-bond futures contracts stops while delivery, based on the last settlement price, remains possible according to the delivery sequence. The so-called wild card play (or end-of-the day option or six hours option) and the end-of-month option refer respectively to the timing option during the three day delivery sequence and to the end-of-month rule.

The modeling and measurement of the delivery options implicit in T-bond futures contracts has been extensively examined in the literature. In particular, the issue of the poor performance of the current conversion factor system (which is directly related to the value of the quality option) has been the subject of a substantial volume of research (see for instance Kane and Marcus (1984), Kane and Marcus (1986a), Jones (1985), Arak, Goodman and Ross (1986), Johnston and McConnel (1989) and Schulte and Violi (2001)). More recently, Oviedo (2006) proposed an alternative method for computing futures invoice prices, called the True Notional Bond System (TNBS), aiming to improve the design of the T-bond futures by better achieving the objective of delivering same quality bonds. The author performs an empirical comparison of the losses resulting from delivering alternative bonds (other than the CTD) computed for each of the systems (the CFS and the TNBS). His results show that the average loss in the CFS is more than twice the one of the TNBS.

The main purpose of this paper is to obtain precise values for the quality and timing options under the TNBS and to compare them with the corresponding under the CFS, in a stochastic interest rate framework. Our pricing procedure is an adaptation of the Dynamic Programming (DP) algorithm described in Ben-Abdallah et al. (2007), giving the value of the futures contract under the TNBS as a function of time and current short-term interest rate.

The paper is organized as follows. In Section 2, we give a general description of the TNBS and present the pricing model and the DP formulation. Section 3 describes in details the numerical procedure. In Section 4, we report on some numerical results obtained under the Vasicek (1977) and Cox, Ingersoll and Ross (1985) (hereafter CIR) models for the short rate process. Section 5 is a conclusion.

2 Model and DP formulation

2.1 Notation

We consider a frictionless cash and T-bond futures market in which trading takes place continuously. Denote

- $(c, M) \in \Theta$ an eligible T-bond with a principal of 1 dollar, a continuous coupon rate c , and a maturity M , where the set Θ of eligible bonds is known at the date the contract is written;
- $\{r_t\}$ a Markov process for the risk-free short-term interest rate;
- $\rho(r, t, \tau)$ the price at t of a zero-coupon bond maturing at $\tau \geq t$ when $r_t = r$ under the process $\{r_t\}$

$$\rho(r, t, \tau) = E \left[\exp\left(-\int_t^\tau r_u du\right) \mid r_t = r \right]; \quad (1)$$

- $p(t, c, M, r)$ the price at t of the eligible T-bond (c, M) when $r_t = r$ under the process $\{r_t\}$

$$p(t, c, M, r) = c \int_t^M \rho(r, t, u) du + \rho(r, t, M); \quad (2)$$

- $PV(t, c, M, r)$ the price at t of the eligible T-bond (c, M) when its yield to maturity is r

$$PV(t, c, M, r) = c \int_t^M \exp(-r(u-t)) du + \exp(-r(M-t)). \quad (3)$$

2.2 A general description of the True Notional Bond System

We recall here the description of the TNBS as proposed by Oviedo (2006). This system relies on the criterion of closeness of the futures invoice prices to spot market prices at expiration. In the TNBS, the futures invoice price is set as the present value of the remaining cash flows of the bond to be delivered, discounted at the yield to maturity implied by the settlement price.

In the TNBS, upon delivery at t , if the level of interest rates is $r_t = r$, the futures invoice price of the bond (c, M) is computed in two steps.

First, given the futures settlement price g^* for the T-bond futures, one can compute the yield to maturity of the notional bond, denoted here by y , that makes its price equal to g^* , that is

$$g^* = PV(t, 6\%, 20, y). \quad (4)$$

Second, the futures invoice price of the bond (c, M) is obtained by using rate y to discount its cash flows, that is, the futures invoice price is $PV(t, c, M, y)$.

The CFS and the TNBS use the same inputs to compute the futures invoice price, namely the futures settlement price as well as the characteristics of the bond to be delivered. However, the functional form of the futures invoice price in the TNBS makes all deliverable bonds equal for any level of flat yield curves, while in the CFS this is only achieved at the specific level of 6%.

2.3 Dynamic Programming formulation

To be consistent with the CBOT delivery rules, we consider a sequence of motoring dates t_m^h where the lower index $m = 0, \dots, \bar{n}$ is computed in days from the date the contract is written and the upper index $h \in \{2, 5, 8\}$ indicates the time in hours within that day. Assuming that the contract is written at t_0^2 , we denote the marking to market dates by t_m^2 for $m = 0, \dots, n$, where t_n represents the last trading date during the delivery month. We denote the delivery position dates by t_m^8 for $m = \underline{n}, \dots, \bar{n}$, where $t_{\underline{n}}$ and $t_{\bar{n}}$ are respectively the first and the last date of the delivery month, $0 < \underline{n} < n < \bar{n}$. Finally, the delivery notice dates are denoted t_m^5

for $m = \underline{n} + 1, \dots, \bar{n} + 1$.

Since under the TNBS equation (4) defines an equivalence between the futures settlement price and the implied yield y , we use this yield as a state variable. Our dynamic program is defined on the state space $\{(r, y) : r \geq 0, y \geq 0\}$ and determines the value of the contract for the short trader at each monitoring date, as a function of the spot interest rate at the current date and the yield to maturity of the notional bond as implied by the last settlement price, assuming that the short trader behaves optimally. A fair settlement price makes the value of the contract null for both parties at the settlement dates.

The contract is evaluated by backward recursion in three distinct periods: The end-of-the-month period, where no trading takes place, but delivery is still possible ($m = n, \dots, \bar{n}$), the beginning of the delivery month where trading and delivery are both possible ($m = \underline{n}, \dots, n$), and the period before the delivery month, where no action is taken by the short trader, but the settlement price is adjusted every day ($m = 0, \dots, \underline{n}$).

2.3.1 End-of-the-month Period

If the short trader decides to deliver at a given position date, the expected exercise value $v_m^e(r, y)$ at the delivery position date t_m^8 and the actual exercise value $v_m^a(r, y)$ for the short trader, for $m = n, \dots, \bar{n}$, are expressed as follows:

$$v_m^e(r, y) = E \left[\left(v_m^a \left(r_{t_{m+1}^5}, y \right) e^{-\int_{t_m^8}^{t_{m+1}^5} r_u du} \right) \mid r_{t_m^8} = r \right], \quad (5)$$

$$v_m^a(r, y) = \max_{(c, M) \in \Theta} \{ PV(t_{m+1}^5, c, M, y) - p(t_{m+1}^5, c, M, r) \}, \quad (6)$$

where (5) represents expectation, discounted over the time period between the position and notice dates, of the exercise value (6) for the short trader, that is, the difference between the futures invoice price and the price of the cheapest-to-deliver bond.

Otherwise, if the short trader decides not to deliver at t_m^8 , for $m = n, \dots, \bar{n}$, the holding value $v_m^h(r, y)$ is computed by no-arbitrage to be the expected value of the future potentialities of the contract and given by (9) below. The short trader will of course issue an intention to

deliver at (t_m^8, r, y) if and only if

$$v_m^e(r, y) > v_m^h(r, y). \quad (7)$$

The value function for the short trader at t_m^8 , for $m = n, \dots, \bar{n}$, is thus defined recursively by:

$$v_m^8(r, y) = \max \left\{ v_m^e(r, y), v_m^h(r, y) \right\}, \quad (8)$$

$$v_m^h(r, y) = E \left[v_{m+1}^8 \left(r_{t_{m+1}^8}, y \right) e^{-\int_{t_m^8}^{t_{m+1}^8} r_u du} \mid r_{t_m^8} = r \right], \quad (9)$$

$$v_{\bar{n}}^8(r, y) = v_{\bar{n}}^e(r, y), \quad (10)$$

and the settlement value for the short trader at $(t_n^2, r_{t_n^2}, y)$ is the expected discounted value at $t_{m'}^8$, where $m' = n$:

$$v_{m'}^2(r, y) = E \left[v_{m'}^8 \left(r_{t_{m'}^8}, y \right) e^{-\int_{t_{m'}^2}^{t_{m'}^8} r_u du} \mid r_{t_{m'}^2} = r \right]. \quad (11)$$

The fair settlement price $g_n^*(r)$ at t_n^2 is such that the value to both parties is 0, taking into account the timing and quality options. To obtain it, a rate y^* is selected such that $v_n^2(r, y^*) = 0$ for all r , thus obtaining a function $y^*(r)$ at t_n^2 ; the fair settlement price $g_n^*(r)$ at t_n^2 is the present value of the notional bond (6%, 20) when its future cash flows are discounted at the rate $y^*(r)$.

2.3.2 Delivery Month

During the delivery month, the exercise value functions $v_m^e(r, y)$ and $v_m^a(r, y)$ at respectively t_m^8 and t_m^5 , for $m = \underline{n}, \dots, n - 1$ remain the same as in the end-of-the-month period and are given by (5)-(6), while the holding value at t_m^8 accounts for the interim payment at the next

marking-to-market date, that is,

$$\begin{aligned}
v_m^h(r, y) &= E \left[\left(PV(t_m^2, 6\%, 20, y) - g_{m+1}^*(r_{t_{m+1}^2}) \right) e^{-\int_{t_m^8}^{t_{m+1}^2} r_u du} \right. \\
&\quad \left. + v_{m+1}^2(r_{t_{m+1}^2}) e^{-\int_{t_m^8}^{t_{m+1}^2} r_u du} \mid r_{t_m^8} = r \right] \\
&= E \left[\left(PV(t_m^2, 6\%, 20, y) - g_{m+1}^*(r_{t_{m+1}^2}) \right) e^{-\int_{t_m^8}^{t_{m+1}^2} r_u du} \mid r_{t_m^8} = r \right]. \tag{12}
\end{aligned}$$

The value function at t_m^8 and t_m^2 is then given by (8) and by (11), with $m' = m$.

2.3.3 Initial Period

Within the time period $[t_0, t_{\underline{n}-1}^2]$, delivery is not possible, so that the value of the contract for the short trader only involves taking into account the interim payments in the marking to market account. The value function at t_m^2 , for $m = 0, \dots, \underline{n} - 1$, is thus given by

$$v_m^2(r) = E \left[\left(g_m^*(r) - g_{m+1}^*(r_{t_{m+1}^2}) \right) e^{-\int_{t_m^2}^{t_{m+1}^2} r_u du} \mid r_{t_m^2} = r \right] = 0. \tag{13}$$

Therefore, the successive settlement prices can be obtained by the recursive relation

$$g_m^*(r) = \frac{E \left[g_{m+1}^*(r_{t_{m+1}^2}) e^{-\int_{t_m^2}^{t_{m+1}^2} r_u du} \mid r_{t_m^2} = r \right]}{\rho(r, t_m^2, t_{m+1}^2)} \text{ for all } r, m = 0, \dots, \underline{n} - 1. \tag{14}$$

3 The Dynamic Programming procedure

Equations (5)-(14) define a dynamic program that can be used to find the fair settlement prices and the optimal timing and choosing strategies for the short trader by backward induction. In this section, we describe a numerical procedure to solve this dynamic program, which does not admit a closed-form solution, even for the most simple case where the interest rate for all maturities is assumed to be constant. Two specific numerical problems must be addressed.

First, the optimization in (6) which consists in finding the CTD by solving the following

expression at t_{m+1}^5

$$\max_{(c,M) \in \Theta} \left\{ PV(t, c, M, y) - \left(c \int_t^M \rho(r, t, u) du + \rho(r, t, M) \right) \right\}. \quad (15)$$

It is easy to see that the optimal coupon is extremal and given by either $\underline{c} \equiv \min c$ or $\bar{c} \equiv \max c$. Since the set of eligible bonds is fixed, a simple enumeration of eligible bonds with the extremal coupons will yield the CTD and the value of $v_m^a(r, y)$.

The second problem is the computation of the expectations in (5), (9), (11), (12) and (14) of functions which are analytically intractable. To solve it, we compute expectations of linear finite elements interpolation functions over a finite discretization grid.

Let $G_1 = \{a_1, \dots, a_q\}$ be a grid defined on the set of interest rates, with the convention that $a_0 = -\infty$ and $a_{q+1} = +\infty$. Given a function $h : G_1 \rightarrow \mathbb{R}$, the interpolation function $\hat{h} : \mathbb{R} \rightarrow \mathbb{R}$ is given by:

$$\hat{h}(r) = \sum_{i=0}^q (\alpha_i + \beta_i r) I(a_i \leq r < a_{i+1}), \text{ for all } r \in \mathbb{R}, \quad (16)$$

where the function I is the indicator function and the coefficients α_i and β_i are obtained by matching \hat{h} and h on G_1 and extrapolating outside of G_1 , that is

$$\alpha_i = \frac{a_{i+1}h(a_i) - a_i h(a_{i+1})}{a_{i+1} - a_i}, \quad (17)$$

$$\beta_i = \frac{h(a_{i+1}) - h(a_i)}{a_{i+1} - a_i}, i = 1, \dots, q-1, \quad (18)$$

and $\alpha_0 = \alpha_1, \beta_0 = \beta_1, \alpha_q = \alpha_{q-1}, \beta_q = \beta_{q-1}$.

The expected value at t and $r_t = a_k$ of a future payoff $\hat{h}(\cdot)$ at τ is then given by:

$$\begin{aligned} \tilde{h}(t, \tau, a_k) &\equiv E \left[\hat{h}(r_\tau) e^{-\int_t^\tau r_u du} \mid r_t = a_k \right] \\ &= E \left[\sum_{i=0}^q (\alpha_i + \beta_i r_\tau) I(a_i \leq r_\tau < a_{i+1}) e^{-\int_t^\tau r_u du} \mid r_t = a_k \right] \\ &= \sum_{i=0}^q \alpha_i A_{k,i}^{t,\tau} + \beta_i B_{k,i}^{t,\tau} \text{ for all } a_k \in \mathcal{G}_1, \end{aligned} \quad (19)$$

where $A_{k,i}^{t,\tau}$ and $B_{k,i}^{t,\tau}$ are transition parameters given by the following expressions:

$$A_{k,i}^{t,\tau} \equiv E \left[I(a_i \leq r_\tau < a_{i+1}) e^{-\int_t^\tau r_u du} \mid r_t = a_k \right] \quad (20)$$

and

$$B_{k,i}^{t,\tau} \equiv E \left[r_\tau I(a_i \leq r_\tau < a_{i+1}) e^{-\int_t^\tau r_u du} \mid r_t = a_k \right], \quad (21)$$

where $t_0 \leq t \leq \tau$, $k = 1, \dots, q$, and $i = 0, \dots, q$.

We assume that these transition parameters and the discount factor $\rho(r, t, \tau)$ can be obtained with precision from the dynamics of $\{r_t, t \geq t_0\}$. Notice that for several dynamics of the interest rates, closed-form solutions exist for the transition parameters and discount factor, as discussed in Ben-Ameur et al. (2007). Examples include Vasicek (1977), CIR (1985), and Hull and White (1990). Closed-form formulas for the transition parameters and discount factor for the Vasicek and the CIR models are recalled in the Appendix.

We also define the grid $G_2 = \{y_1, \dots, y_q\}$ on the set of yields to maturity of the notional bond, where y_1 and y_q are computed such that $p(t_0^2, 6\%, 20, a_1) = PV(t_0^2, 6\%, 20, y_1)$ and $p(t_0^2, 6\%, 20, a_q) = PV(t_0^2, 6\%, 20, y_q)$.

The algorithm consists in solving the dynamic program (5)-(14) by backward induction from the last delivery position date $t_n^{\frac{8}{n}}$ on the grid $G_1 \times G_2$.

We start by finding the CTD and the actual exercise values for the short trader at the notice dates on all the points of $G_1 \times G_2$, that is, for all possible interest rates and for all possible futures prices represented by the yield to maturity of the notional bond. For each yield to maturity of the notional bond, a linear interpolation produces the actual exercise value as a continuous function of the interest rate. The expected exercise values are then obtained at the position date. These are compared with the holding values, which are known on the two-dimensional grid. Again, for a given yield to maturity, the optimal value function at the position date is interpolated and the expectation is computed between either two successive position dates (during the end-of-the-month period) or the last settlement and current position dates (during the delivery month). This produces the value for the short trader at the settlement date as a function of the interest rate and the yield to maturity of

the notional bond, which is null for a fair settlement price. At every settlement date, a simple search on the grid G_2 for a given interest rate gives the yield to maturity of the notional bond $y^*(r)$ that makes the settlement value null for that interest rate. This rate is then used to discount the notional bond's cash flows, thus obtaining the fair futures price as a function of spot interest rate. These are then interpolated, and their expected values computed, in order to obtain the holding value of the futures contract.

The detailed algorithm is available upon request.

4 Numerical Illustration

In our numerical experiments, the finite set of deliverable bonds contains 62 bonds with maturity ranging from $\underline{M} = 15$ years to $\overline{M} = 30$ years in steps of 6 months. Since only the bonds with extremal coupon rates are optimal to deliver, we consider only two coupon rates corresponding to the highest and lowest coupon rates in the current CBOT set of deliverable bonds, namely $\bar{c} = 7.625\%$ and $\underline{c} = 4.5\%$. The inception date is chosen to be three months prior to the first day of the delivery month.

We apply our dynamic programming procedure to obtain futures prices at the inception date under both the Vasicek and CIR term structure models given respectively in (25) and (33), using the closed-form formulas (28)-(30) or (38)-(40) for the discount factor and transition parameters. Table 1 below gives the (risk neutral) parameter values used in the numerical experiments, where \bar{r} is the long-term mean, κ is the mean reversion speed and σ is the volatility of the short-term interest rate. These parameters are those of Shoji and Osaki (1996) who estimate these models using the 1-month U.S. Treasury Bill rate over the period 1964-1992.

	Vasicek	CIR
\bar{r}	0.062098	0.061677
κ	0.565888	0.545788
σ	0.025416	0.091471

The interest rates grid points a_1, \dots, a_q are selected to be equally spaced with $a_0 = -\infty$,

$a_1 = \bar{r} - 8\sigma d^V(t)$, $a_q = \bar{r} + 8\sigma d^V(t)$ for the Vasicek model, while $a_1 = \max(0; \bar{r} - 8\sigma d^C(t))$ and $a_q = \bar{r} + 8\sigma d^C(t)$ for the CIR model, where

$$d^V(t) = \sqrt{\frac{1}{2\kappa}(1 - \exp(-2t\kappa))}, \quad (22)$$

$$d^C(t) = \sqrt{\frac{\bar{r}}{\kappa}(\exp(-t\kappa) - \exp(-2t\kappa)) + \frac{\bar{r}}{2\kappa}(1 - \exp(-t\kappa))^2} \quad (23)$$

and t is in years (for an inception date of 3 months, $t = 0.25$). The number of grid points is 600.

We disentangle the individual effects of each implicit option by pricing four futures contracts embedding different combinations of these options, namely

- F1 : the straight futures contract offering no options at all and corresponding to the case where the short trader declares his intention to deliver on the first position day of the delivery month and delivers the notional bond,
- F2 : the contract offering the quality option alone, where the short trader chooses on day $\underline{n}+1$ the bond to be delivered among the deliverable basket,
- F3 : the contract offering only the timing option, allowing the short trader to deliver the notional bond anytime during the delivery month according to the delivery sequence, and
- F4 : the full contract offering all the embedded delivery options to the short trader.

The computation of these four prices allows us to price each option alone as well as in the presence of the other option. For instance, we compute the following differences

- F1-F4 is the value of all the embedded options,
- F3-F4 gives the value of the quality option in the presence of the timing option,
- F1-F2 is the value of the quality option without timing,
- F1-F3 is the value of the timing option without quality and

- F2-F4 is the value of the timing option when the quality option is offered to the short trader.

Notice that definitions of implicit delivery options are not uniform throughout the literature, and one must be cautious in comparing results across studies. According to our definition, the timing option gives the short trader the right to deliver late on any day during the delivery month. Some papers define the timing option as the option to deliver early in the delivery month. The small value they obtain can be explained by the fact that delaying delivery is often optimal.

4.1 Option Prices Under TNBS vs CFS

We now report on the prices of the quality and timing options at the inception date for levels of interest rates ranging from 3.5% to 8.5%. Parameter values are those given in Table 1.

Figure 1 compares the values of the embedded quality option at the inception date (with and without the timing option) as a function of the interest rate for the Vasicek and CIR dynamics. We notice from Figure 1 that the value of the quality option with respect to interest rates shows a minimum at the level of the long-run mean. Recall that, under the TNBS, bonds are equal for delivery for any level of flat yield curves. Under stochastic interest rates evolving according to a mean reverting process such as the Vasicek and CIR dynamics, the yield curve will be closer to a nearly flat yield curve if the current spot interest rate is in the neighborhood of the long-term mean level. Therefore, at this level of interest rates, homogeneity of the basket of deliverable bonds reaches its best level.

[Insert Figure 1 here]

Without the timing option, the quality option is worth an average 0.47 percentage points of par (ppp) for the Vasicek model while this value is 0.42 ppp for the CIR model. When the timing option is embedded in the contract, we notice a slight increase in the value of the quality option, which is worth an average 0.5 ppp (Vasicek) and 0.45 ppp (CIR). The fact that the quality option is more valuable in the presence of the timing option is due to the

interaction and the interdependence that exist between these two options. We also notice from Figure 1 that the difference between the quality option with and without the timing option reaches its maximum at the level of the long-term mean, indicating that the timing option is the most important at this level.

In Figure 2, we report on the evolution, during the delivery month, of the quality option values under the Vasicek and the CIR dynamics. We observe that the value of the quality option decreases with elapsed time, especially for levels of interest rates in the neighborhood of the long-term mean. This is expected since uncertainty about the CTD is reduced as we approach the last day of the delivery month, and so should be the value of the quality option.

[Insert Figure 2 here]

Figure 3 plots the value of the timing option at the inception date (with and without the quality option) for both dynamics considered here.

[Insert Figure 3 here]

Without the quality option, the timing option is worth an average 0.24 ppp (Vasicek) and 0.29 ppp (CIR) while when the quality option is offered to the seller, the timing option is more valuable and is worth an average 0.28 ppp (Vasicek) and 0.32 ppp (CIR). Again, because of the interaction between the options, the timing option is generally more valuable in the presence of the quality option.

We now move to a comparison of the values of the quality and timing options obtained under the conversion factor system and the TNBS. In general, for various parameter values, we verified that the quality option becomes more valuable under CFS than TNBS when the long-term mean moves away from the notional rate, and that the timing option can be either higher or lower under TNBS than under CFS (see for instance Figures 4-6 comparing option values at inception under the case where the long-run mean is at the level of 3% while the other parameter values are those given in Table 1). However, when the long-run mean is around the notional rate, TNBS is no longer better. This is illustrated in Figures 7-9

comparing the quality option, the timing option as well as all the options together under both systems at inception (i.e., three months prior to the first delivery day) for case Table 1. We notice from these figures that all the embedded options are more valuable under the TNBS at inception. It is worthwhile mentioning here that this numerical illustration corresponds to the situation where the CFS achieves its best performance in terms of relating the deliverable bonds to the reference bond, which can explain the outperformance of this system compared to the TNBS. For details about the valuation of the quality and timing options under the CFS, we refer to Ben-Abdallah et al. (2007).

[Insert Figures 4-9 here]

We present in figures 10-15 the results for the sensitivity of the option values to the parameters of the interest rate models at inception and for both dynamics considered in this paper. The base case parameters values are $\bar{r} = 0.06$, $\kappa = 0.5$ and $\sigma = 0.02$ for the Vasicek dynamics and $\bar{r} = 0.06$, $\kappa = 0.5$ and $\sigma = 0.09$ for the CIR dynamics.

From these figures, we observe that both the quality and timing options values are negatively affected by the mean reversion speed. This can be explained by the fact that an increase in the mean reversion rate, which determines the relative volatilities of long and short rates, dampens out short-term rate movements quickly and therefore reduces the long-term volatility, which is positively related to the quality option value. We also observe that the relation between the long-term mean and the quality or timing options could be either positive or negative depending on the level of spot interest rates. The timing option is however observed to be always positively affected by the long-term mean. Furthermore, as expected, the relation between the quality option value and the volatility is observed to be positive. In addition, the hump in the curve representing the timing option observed in the neighborhood of the long-run mean is more noticeable for curves associated with small levels of long-term mean or speed of adjustment.

It is worthwhile mentioning here that the behavior of the embedded delivery options under the TNBS does not show irregularities with respect to interest rates or unexpected relations

with respect to the input parameters of the interest rate process as it is observed under the CFS. For more details about the sensitivity of options values under the CFS, we refer to Ben-Abdallah et al. (2007).

[Insert Figures 10-15 here]

4.2 Optimal delivery strategy

In this section, we examine the optimal delivery strategy during the delivery month under the Vasicek model. Since the decision whether to deliver or not is made on the position day at 8:00 p.m. while the choice of the bond to be delivered is made the next day at 5:00 p.m., results about the decision and the CTD are presented separately. We assume 22 business days in the delivery month and we choose to report on the optimal decision (the CTD) on days 15 (16), 18 (19) and 21 (22). For this illustration, we use the set of parameters given in Table 1.

The optimal decision on the position day as well as the identification of the CTD on the notice day depend on the two state variables, namely the current short-term interest rate r and the yield to maturity of the notional bond y implied by the last futures settlement price.

Figure 16 presents the optimal decision on day 15 of the delivery month (the last futures trading day) for possible values of the two state variables. Notice that only a small area around the diagonal of Figure 16 is likely to be observed, corresponding to possible variations of the rate in the 6 hours period separating settlement and position times. This area is presented in Figure 17 and we notice that early exercise can be optimal during the delivery month. Figure 18 represents the optimal decision 4 days before the end of the delivery month and shows that exercising the end-of-the-month option can be optimal. Also, according to this figure, if the short trader has decided to make delivery on day 18 of the delivery month, the CTD on the next day is the bond $(\underline{c}, \underline{M})$, corresponding to both minimal coupon rate and maturity (such a bond is not CTD for deterministic rates). Finally, we present in Figure 19 the cheapest-to-deliver bond on the last day of the delivery month (day 22) for various possible combinations of r and y . Similar results about the delivery strategy are obtained

under the CIR dynamics.

[Insert Figures 16-19 here]

5 Conclusion

In this paper, we propose an efficient numerical method for pricing CBOT T-bond futures when the TNBS is the system adopted to obtain same quality bonds. This method takes into account all the inter-dependent embedded delivery options. We price the contract and the delivery options in a stochastic interest rate framework. Numerical illustrations, provided here under the Vasicek and CIR models, show that, in a stochastic framework, TNBS does not always outperform the CFS. However, as the long-term mean moves away from the level of the notional rate, the TNBS performs increasingly better than the CFS.

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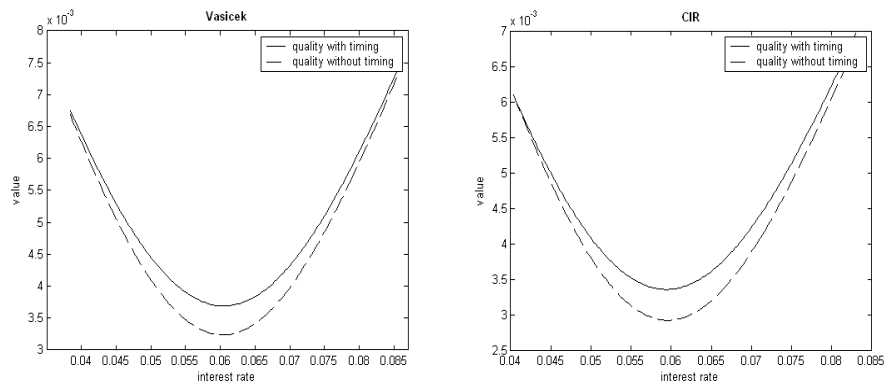


Figure 1: Quality option values vs interest rates at inception

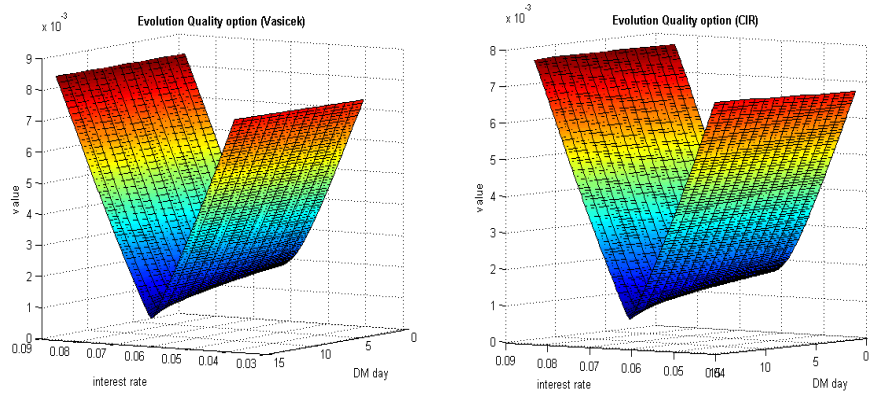


Figure 2: Evolution of the value of the quality option through the DM

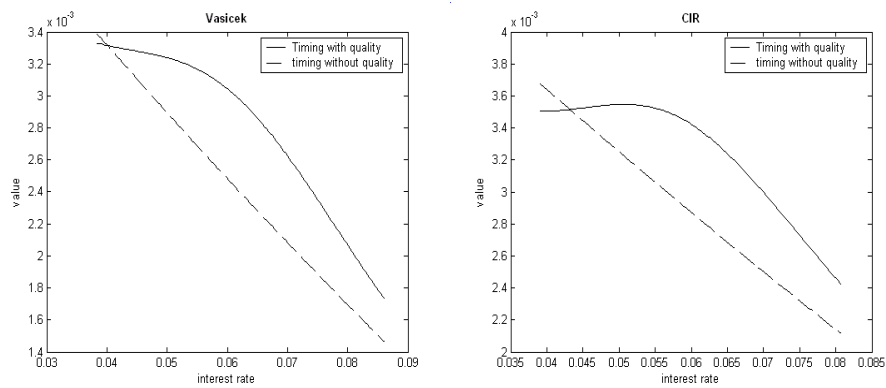


Figure 3: Timing option values vs interest rates at inception

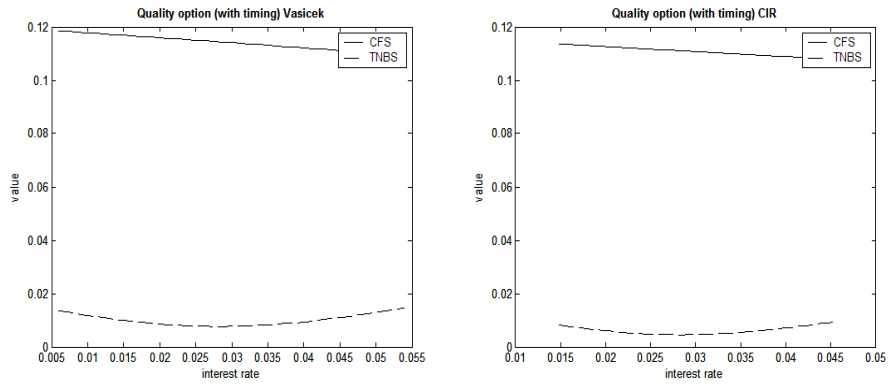


Figure 4: Comparison of the quality option values under the CFS and the TNBS at inception ($r_{bar}=3\%$)

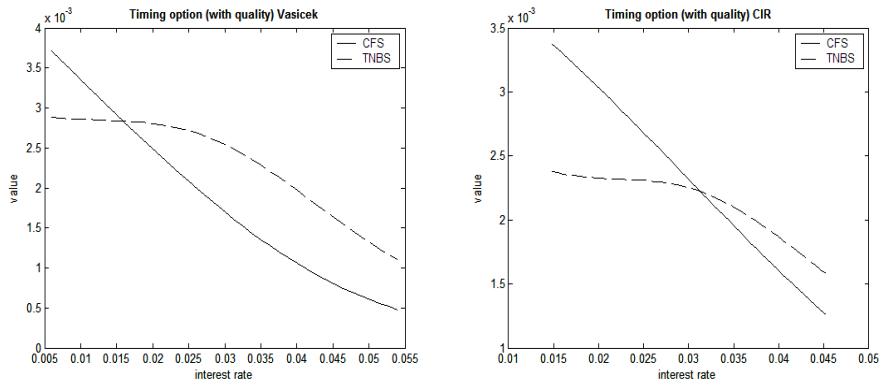


Figure 5: Comparison of the timing options values under the CFS and the TNBS at inception ($r_{bar}=3\%$)

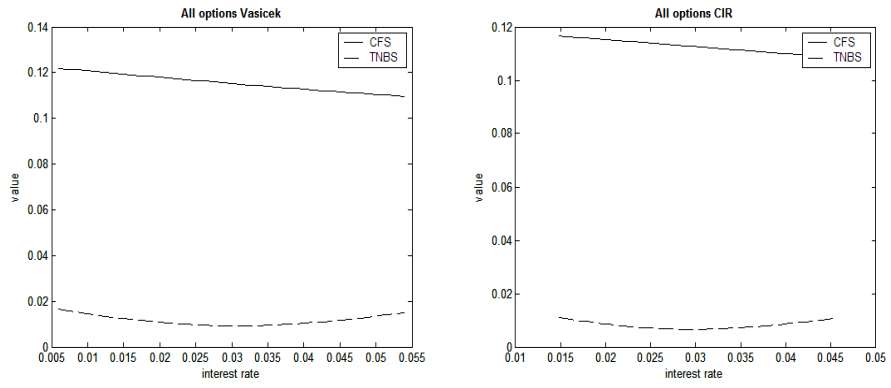


Figure 6: Comparison of all the options under the CFS and the TNBS at inception ($\bar{r}=3\%$)

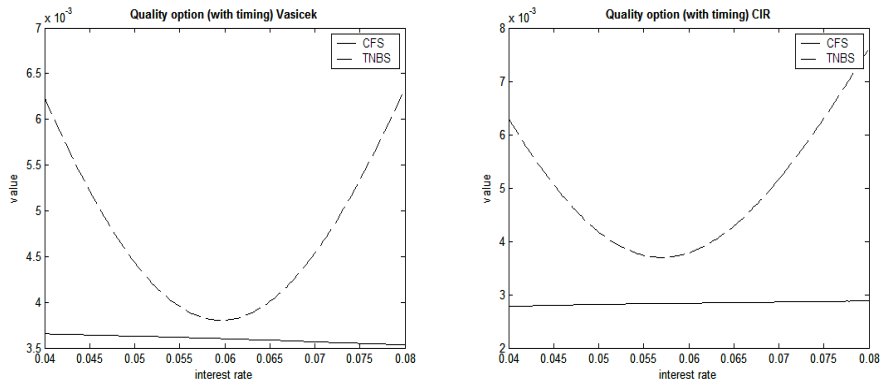


Figure 7: Comparison of the quality option values under the CFS and the TNBS at inception ($\bar{r}=6\%$)

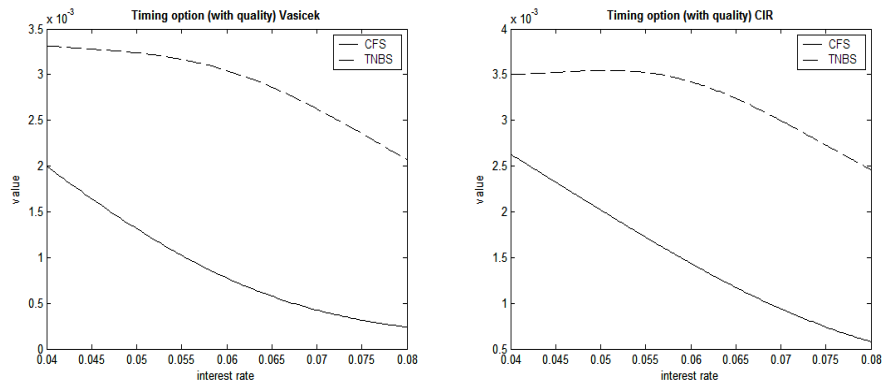


Figure 8: Comparison of the timing options values under the CFS and the TNBS at inception ($\bar{r}=6\%$)

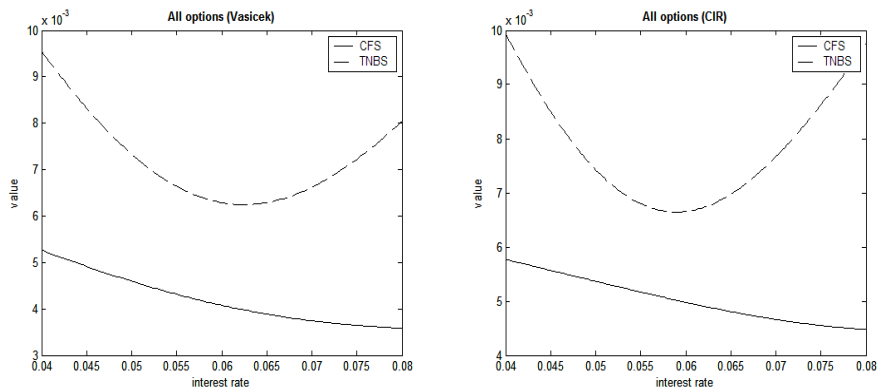


Figure 9: Comparison of all the options under the CFS and the TNBS at inception ($\bar{r}=6\%$)

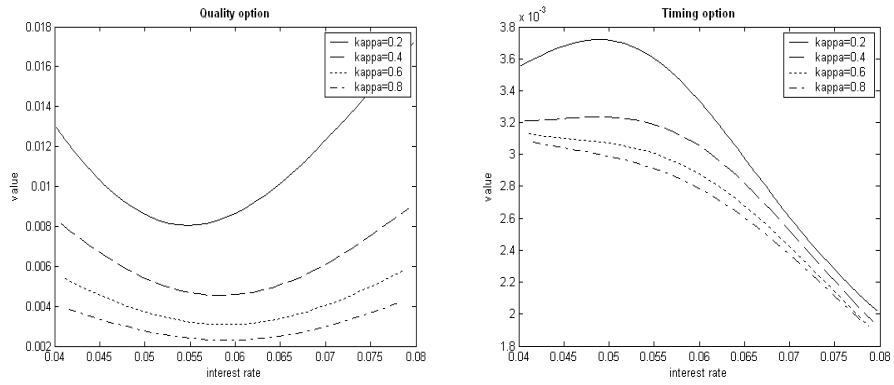


Figure 10: Options values sensitivities to kappa (Vasicek)

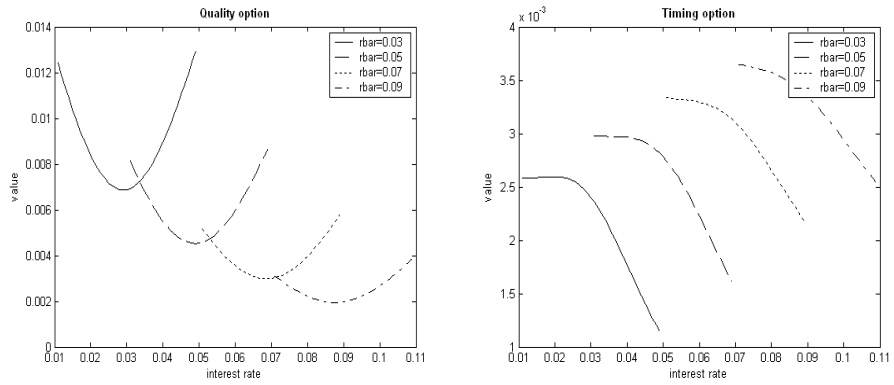


Figure 11: Options values sensitivities to rbar (Vasicek)

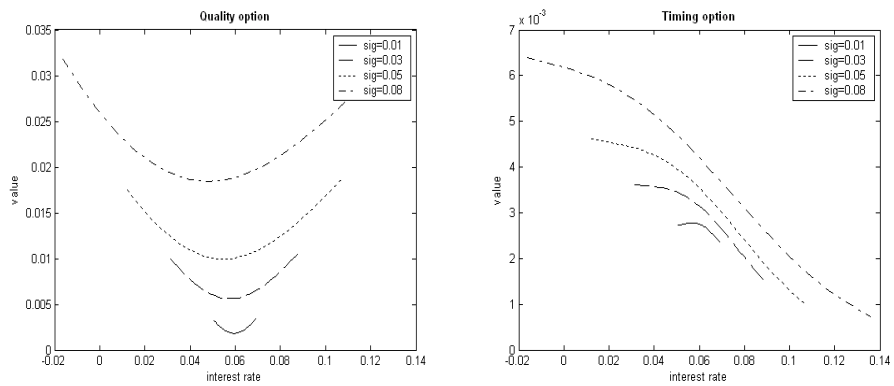


Figure 12: Options values sensitivities to sigma (Vasicek)

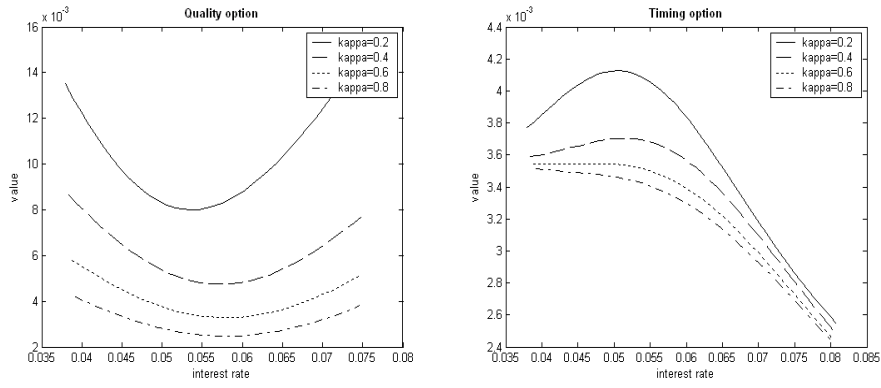


Figure 13: Options values sensitivities to kappa (CIR)

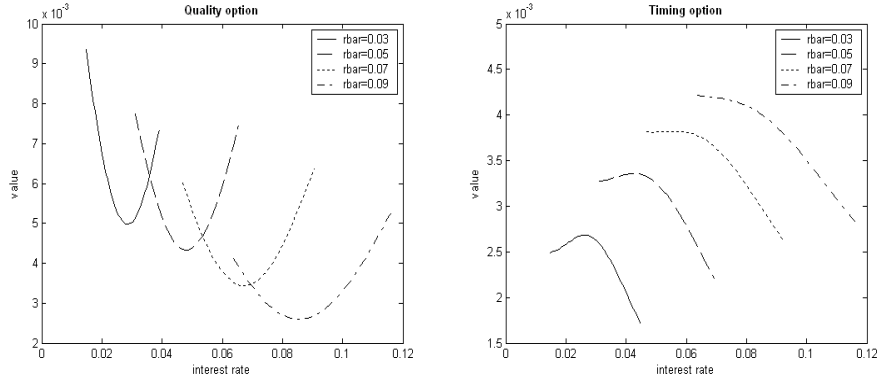


Figure 14: Options values sensitivities to rbar (CIR)

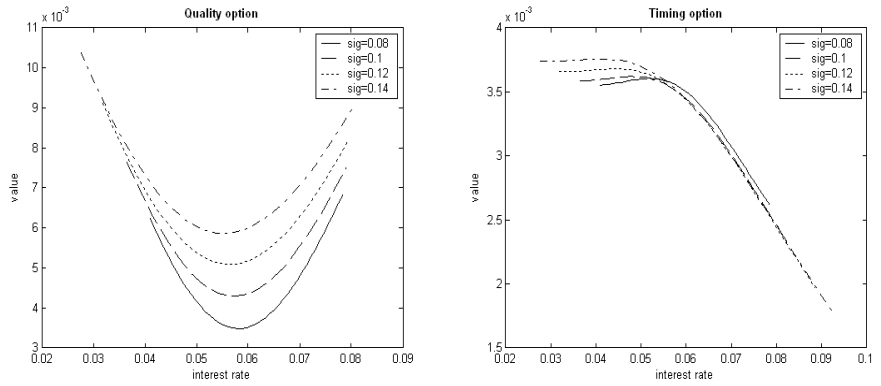


Figure 15: Options values sensitivities to sigma (CIR)

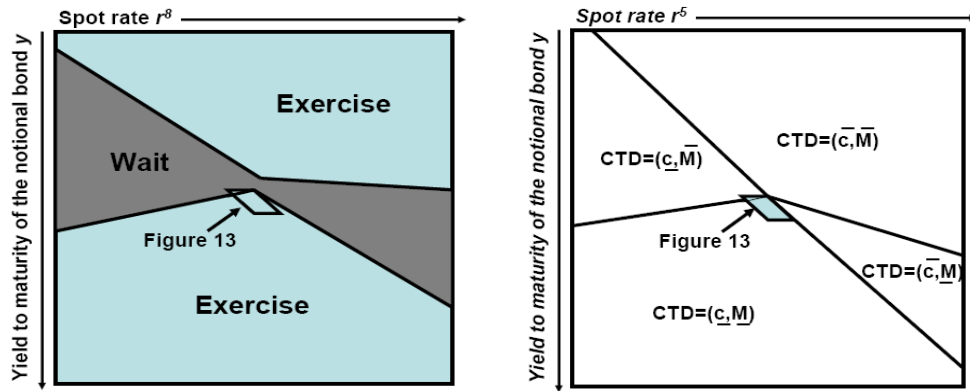


Figure 16: Optimal decision and CTD on days 15 and 16 of the DM (Vasicek)

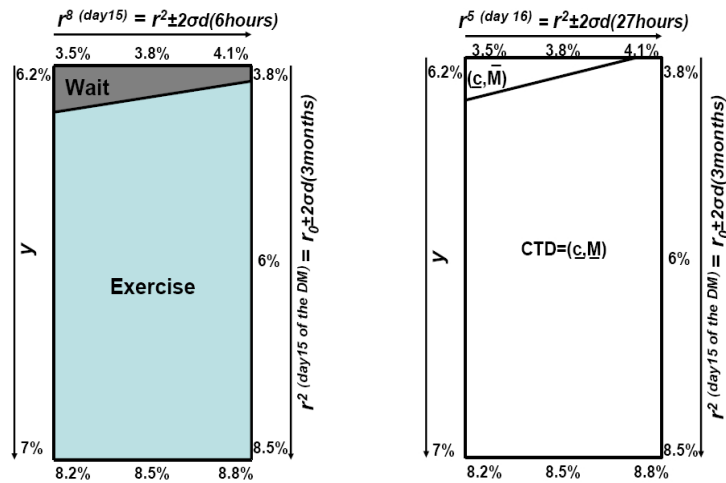


Figure 17: Optimal decision and CTD on days 15 and 16 of the DM (Vasicek)

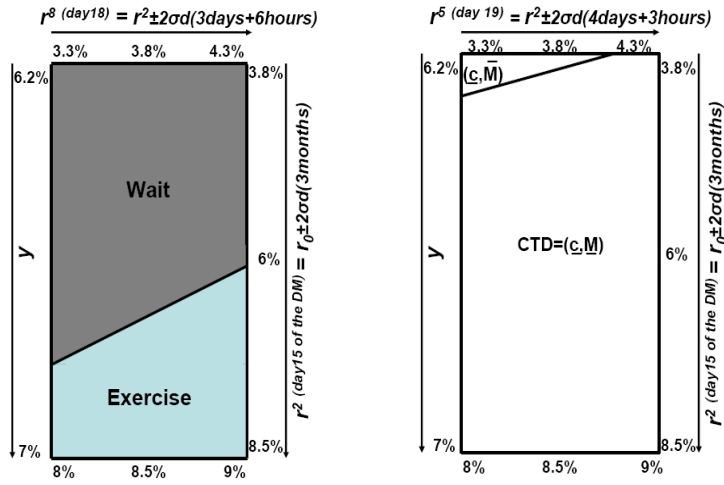


Figure 18: Optimal decision and CTD on days 18 and 19 of the DM (Vasicek)

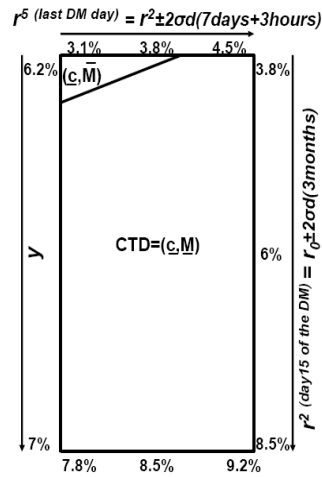


Figure 19: CTD on the last day of the DM (Vasicek)

6 Appendix

6.1 Transition parameters

We give, for the Vasicek (1977) and CIR (1985) models, the closed-form formulas for the transition parameters $A_{k,i}^{t,\tau} \equiv A_{k,i}^\delta$ and $B_{k,i}^{t,\tau} \equiv B_{k,i}^\delta$ ($\delta = \tau - t$) defined respectively in (20) and (21) as well as for the discount factor $\rho(r, t, t + \delta)$ defined in (1). For both models, the derivation of these closed-forms starts from the distribution of the random vector

$$\left(r_{t+\delta}, \int_t^{t+\delta} r_u du \right) \quad (24)$$

conditional on the value of r_t , for $0 \leq t \leq t + \delta$. For proofs and more details about the derivation of these closed-forms we refer to Ben-Ameur et al. (2007).

6.1.1 The Vasicek model

Under the risk-neutral probability measure, the interest rate process is the solution to the following stochastic differential equation

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dB_t, \quad \text{for } t \geq 0, \quad (25)$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion, κ is the mean reversion speed, \bar{r} is the long-term mean and σ is the volatility. For the Vasicek model, the distribution of the random vector (24) conditional on $r_t = r$ is bivariate normal with mean

$$\mu(r, \delta) = (\mu_1(r, \delta), \mu_2(r, \delta)) = \left(\bar{r} + e^{-\kappa\delta}(r - \bar{r}), \bar{r}\delta + \frac{1 - e^{-\kappa\delta}}{\kappa}(r - \bar{r}) \right) \quad (26)$$

and covariance matrix

$$\Sigma(\delta) = \begin{bmatrix} \sigma_1^2(\delta) & \sigma_{12}(\delta) \\ \sigma_{21}(\delta) & \sigma_2^2(\delta) \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\delta}) & \frac{\sigma^2}{2\kappa^2}(1 - 2e^{-\kappa\delta} + e^{-2\kappa\delta}) \\ \sigma_{21} & \frac{\sigma^2}{2\kappa^3}(-3 + 2\kappa\delta + 4e^{-\kappa\delta} - e^{-2\kappa\delta}) \end{bmatrix}. \quad (27)$$

The discount factor and the transition parameters are then given by

$$\rho(r, t, t + \delta) = \exp(-\mu_2(r, \delta) + \sigma_2^2(\delta)/2), \quad (28)$$

$$A_{k,i}^\delta = e^{-(\mu_2(a_k, \delta) + \sigma_2^2(\delta)/2)} [\Phi(x_{k,i}) - \Phi(x_{k,i-1})], \quad (29)$$

and

$$B_{k,i}^\delta = e^{-(\mu_2(a_k, \delta) + \sigma_2^2(\delta)/2)} [(\mu_1(a_k, \delta) - \sigma_{12}(\delta))(\Phi(x_{k,i}) - \Phi(x_{k,i-1})) - \sigma_1(\delta)(e^{-x_{k,i}^2} - e^{-x_{k,i-1}^2})/\sqrt{2\pi}], \quad (30)$$

where

$$x_{k,i} = (a_i - \mu_1(a_k, \delta) + \sigma_{12}(\delta))/\sigma_1 \text{ for } i = 0, \dots, q, \quad (31)$$

$$x_{k,-1} = -\infty \quad (32)$$

and Φ is the standard normal distribution function.

6.1.2 The CIR model

Under the risk-neutral probability measure, the interest rate process is the solution to the following stochastic differential equation

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dB_t, \text{ for } t \geq 0. \quad (33)$$

For the CIR model, the distribution of the random vector (24) conditional on $r_t = r$ is characterized by its Laplace transform:

$$\begin{aligned} & E \left[\exp(-\omega \int_t^{t+\delta} r_u du - \nu r_{t+\delta}) \mid r_t = r \right] \\ &= \exp(X(\delta, \omega, \nu) - rY(\delta, \omega, \nu)), \end{aligned} \quad (34)$$

where

$$X(\delta, \omega, v) = \frac{2\kappa\bar{r}}{\sigma^2} \log \left[\frac{2\gamma(\omega)e^{(\gamma(\omega)+\kappa)\delta/2}}{(v\sigma^2 + \gamma(\omega) + \kappa)(e^{\gamma(\omega)\delta} - 1) + 2\gamma(\omega)} \right], \quad (35)$$

$$Y(\delta, \omega, v) = \frac{v(\gamma(\omega) + \kappa + e^{\gamma(\omega)\delta}(\gamma(\omega) - \kappa)) + 2\omega(e^{\gamma(\omega)\delta} - 1)}{(v\sigma^2 + \gamma(\omega) + \kappa)(e^{\gamma(\omega)\delta} - 1) + 2\gamma(\omega)} \text{ and} \quad (36)$$

$$\gamma(\omega) = \sqrt{\kappa^2 + 2\omega\sigma^2}. \quad (37)$$

For the CIR model, the discount factor and the transition parameters are given by

$$\rho(r, t, t + \delta) = \exp(X(\delta, 1, 0) - rY(\delta, 1, 0)), \quad (38)$$

$$A_{k,i}^\delta = \rho(a_k, t, t + \delta) \sum_{u=0}^{\infty} e^{-\lambda_k/2 \frac{(\lambda_k/2)^u}{u!}} \left[F_{d+2u}\left(\frac{a_{i+1}}{\eta}\right) - F_{d+2u}\left(\frac{a_i}{\eta}\right) \right], \quad (39)$$

and

$$B_{k,i}^\delta = \rho(a_k, t, t + \delta) \eta \sum_{u=0}^{\infty} e^{-\lambda_k/2 \frac{(\lambda_k/2)^u}{u!}} \left[-2(a_{i+1}f_{d+2u}\left(\frac{a_{i+1}}{\eta}\right) - a_i f_{d+2u}\left(\frac{a_i}{\eta}\right) \right. \\ \left. + (d + 2u)(F_{d+2u}\left(\frac{a_{i+1}}{\eta}\right) - F_{d+2u}\left(\frac{a_i}{\eta}\right)) \right], \quad (40)$$

where F_{d+2u} and f_{d+2u} are the distribution and the density functions of a chi-square random variable with $d + 2u$ degrees of freedom,

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2}, \quad (41)$$

$$\eta = \frac{\sigma^2 (e^{\gamma\delta} - 1)}{2((\gamma + \kappa)(e^{\gamma\delta} - 1) + 2\gamma)}, \quad (42)$$

$$d = \frac{4\kappa\bar{r}}{\sigma^2} \text{ and} \quad (43)$$

$$\lambda_k = \frac{8\gamma^2 e^{\gamma\delta} a_k}{\sigma^2 [(\gamma + \kappa)(e^{\gamma\delta} - 1) + 2\gamma] (e^{\gamma\delta} - 1)}. \quad (44)$$