Time diversification:
Definitions and some closed-form solutions

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Abstract

We establish general conditions under which younger investors should invest a larger proportion of their wealth in risky assets than older ones. In the finite horizon dynamic setting, we show that such phenomenon, known as "time diversification," can occur in the presence of human wealth, guaranteed consumption, or mean-reverting stock returns. We formalize two alternative notions of time diversification commonly confounded in the literature. Analytic solutions are provided for both time-series and cross-sectional forms of time diversification. To our best knowledge, this paper is the first to solve in closed-form the hedging demand for a CARA investor with inter-temporal consumption and a finite horizon, facing mean-reverting expected returns. Our results indicate that horizon can have a significant effect on the portfolio demand of a CARA investor due to inter-temporal hedging.

JEL Classifications: G11; D91

Key words: Portfolio choice; Asset allocation; Time series time diversification; Cross-sectional time diversification.

1. Introduction

Should older investors allocate a smaller proportion of their wealth in risky assets than younger ones? Should you hold a riskier portfolio than your elderly uncle (Samuelson, 1989)? There is a striking difference in opinion between practitioners and academic researchers on these questions. Financial planners almost always advise that older clients invest a smaller proportion of their wealth in stocks than younger clients do.[1] They invoke the doctrine of "time diversification," that the risk of stocks diminishes with the length of the planning horizon. Academic researchers have been skeptical of this conventional wisdom. While a compelling case can be made that stocks outperform bonds over long periods, it does not follow that the proportion of wealth held in risky assets should fall with age. In fact, the canonical models of portfolio choice predict that the portfolio decision should generally be independent of age. Although there are recognized exceptions to this irrelevance proposition, none of them seem to justify the large portfolio adjustments advocated by financial planners. Explaining when and why financial planners may be right about time diversification is a problem that has exercised the ingenuity of financial economists for decades.

Our contribution to this debate is two-fold. First, we distinguish clearly between two different meanings of time-diversification. Understanding the distinction between these concepts goes a long way to resolving the apparent discrepancy between theory and practice. Second, we construct specific examples that allow closed-form solutions in which time diversification emerges in particularly intuitive ways. Although there is a large theoretical literature on time-diversification, there is a paucity of closed-form solutions where it occurs.[2]

There are two notions of time diversification often confounded in the literature. One asserts that an older person should hold a smaller percentage of his wealth in risky assets than a younger person. To adopt Samuelson's (1989, p. 9) evocative image, the idea is that your elderly uncle should hold a less risky portfolio than you do. We call this *cross-sectional diversification*. Notice that this is a ceteris paribus concept, comparing two people who are identical in all respects–including wealth and expected returns–but who differ in their age.

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[1] This advice is ubiquitous. Quinn (1991) and Malkiel (1996) are influential examples.

[2] Since our concern in this paper is theoretical, we do not address the question of whether the risk of stocks declines with the planning horizon empirically. There is a large and contentious literature on this issue. See Bodie (1995), Dempsey, Littler,

This is not what financial consultants have in mind by time diversification. They frequently advise their clients to reduce the holding of risky assets as they get older. We dub this *time-series diversification* since it requires investors to plan on holding a less risky portfolio in the future than today. Unlike cross-sectional diversification, this is not a ceteris paribus concept because it requires investors to anticipate changes in their opportunity set–including wealth and expected returns–over time.

Most of the early academic literature [with the exception of Levy and Spector (1996)] focused on cross-sectional diversification, while the relevant concept for practitioners is clearly time-series diversification. The academic literature over the last decade (which we survey below) has developed models of life-cycle portfolio behavior, and so has provided insights into time-series diversification. Although this has helped to bridge the gap between theory and practice, academics and practitioners were often talking past each other because they were using different notions of time diversification. We show that time-series diversification can occur even in the canonical model. Time-series diversification is an attribute of forecast horizon rather than planning horizon: an investor with an infinite planning horizon may optimally plan to invest less in risky assets five years hence if the expected return is currently high but mean reverts.

When will cross-sectional and time-series diversification occur? In the classic models of portfolio choice, epitomized by Samuelson (1969) and Merton (1971), the answer is that neither should occur. These models predict that the optimal share of wealth invested in risky assets should be independent of the investor's age.[3] The reason is quite simple. In these models, the investor is free to reshuffle his portfolio at each point in time without cost. Applying the Bellman Principle of dynamic programming, the problem of adjusting the portfolio over many periods collapses to the problem of selecting a portfolio once today (given that the investor expects himself to act optimally in the future).

As Jagannathan and Kocherlakota (1996) put it, it is not the length of the planning horizon that matters for portfolio decisions, but how frequently the investor is allowed to adjust his portfolio. It is widely recognized

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and Keasey (1996), Ferguson and Leitiskow (1996), Jagannathan and Kocherlakota (1996), Levy and Spector (1996), Malkiel (1996), Merrill and Thorley (1996,1997), and Oldenkamp and Vorst (1997).

[3] See also Merton and Samuelson (1974) and Samuelson (1990).

that time diversification may emerge as an optimal investment strategy if high transactions costs preclude the investor from revising his portfolio (Levy and Spector, 1996). Explaining why time diversification may be optimal when investors are free to revise their portfolios became something of a theoretical Holy Grail, however. For example, the Nobel-laureate Paul Samuelson (1989, p. 10) confesses to undertaking a ". . . twenty-year quest to justify long-term horizon risk taking . . ."

Samuelson (1994) surveys several theoretical exceptions to the theorem "The Investment Horizon Can Have No Effect on Your Portfolio Proportions."[4]

1. *Human Wealth.* The portfolio decision depends upon total wealth, which consists of financial as well as human wealth (i.e., the present value of the future stream of wage income). For a given stream of wages, however, human wealth would be smaller the fewer the number of years left until retirement. For most people, wage income is nearly risk-free [or at least not correlated with the market, as shown by Heaton and Lucas (1997)]. As one ages, therefore, his/her portfolio should shift away from stocks toward bonds to compensate for the loss in wage income. This result is implicit in Merton (1971), although he did not discuss it. The effect is amplified if the investor can also adjust his supply of labor, since wage income is then negatively correlated with the market (Bodie, Merton, and Samuelson, 1992).
2. *Target Consumption.* Suppose that utility depends on consumption C relative to some target level $\overline{C}$ (as in Stone-Geary preferences): $U\left(C-\overline{C}\right)$. Intuitively, one might think of $\overline{C}$ as representing subsistence consumption or some kind of target consumption–such as college tuition–in mid-life. It turns out that the presence of $\overline{C}$ has exactly the opposite effect of wage income. Intuitively, the target acts as a reduction in wage income. Target consumption therefore makes time diversification occur, but in the wrong direction. It actually requires older people to hold more risky assets than young people.
3. *Mean Reversion.* The traditional portfolio models assume that markets are efficient, so that changes in stock prices are unpredictable. By now, however, it is generally conceded that the logarithms of stock prices have a predictable component (Fama and French, 1988; Poterba and Summers, 1988; Campbell.

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[4] Jagannathan and Kocherlakota's (1996) cover some of the same ground.

Lo, and MacKinlay, 1997)[5]. Intuitively, mean reversion should make stocks less risky the longer the time available for the prices to gravitate towards their long run values (Samuelson, 1988, 1994).

We develop a model that is simple enough to allow a closed-form solution, but captures fairly general conditions under which time diversification can occur in a dynamic portfolio model without transaction costs. The three cases described by Samuelson (1994) emerge as special cases of our model. In addition, our results show that there is something more than mean-reversion at work in Samuelson's (1988) example. If stock prices are not random walks, time diversification can occur through the hedging demand when the change in the expected return is correlated with the current return. This provides another sufficient condition for time diversification to occur.

This feature of the model is related to Kim and Omberg (1996). They address the issue of when a portfolio will be non-myopic, that is, when holding of risky assets depends on the distribution of asset returns in future periods in addition to the current one. However, our objective is to address more specific questions, whose answers require more stringent conditions. We seek particular forms of non-myopic behavior. When will a younger person invest more in risky assets than an older person (cross-sectional diversification)? When does an investor anticipate investing less in risky assets as he ages (time-series diversification)? Kim and Omberg derive conditions for a local form of cross-sectional diversification, while our definition is more general. They assume that the investor only gets utility from wealth in the final period, which is much more restrictive than our model. Moreover, Kim and Omberg do not explicitly address time series diversification.

Recent literature has expanded upon the three cases surveyed by Samuelson (1994) in several directions. Thorley (1995) and Gollier (2002) show that decreasing absolute risk aversion can lead to more conservative portfolios as the investor's horizon shortens. Cocco (2004) reports a crowding out effect from risky housing investment on stock holdings. Heaton and Lucas (1997) and Koo (1999) introduce risky labor income uncorrelated with stocks. Their calibration exercise implies large holdings in stocks. Viceira (2001) demonstrates that positive correlation between labor income and stock returns may reduce agent's investment in stocks as a

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[5] Bali, Demirtas and Levy (2008) discuss non linear mean reversion in stock prices.

hedge against wage risk. In the model of Benzoni et al. (2007), labor income and stock returns are co-integrated, wage behaves "stock-like" at the long horizon, and "bond-like" at the short horizon. They reproduce a hump-shaped demand for stocks. Polkovnichenko (2007) combines habit formation with uninsurable labor income to generate portfolio demand that is non-monotonic in investor's horizon.[6]

Most of these studies [except for Thorley (1995) and Gollier (2002)] resort to numerical methods or approximations such as log-linearization. Closed-form solutions of portfolio choice that exhibit time diversification are few and far between. Our work joins a strand of recent literature that provides analytical solutions to Merton's portfolio choice problem with stochastic investment opportunity set. In Merton's framework, portfolio weight is given as a solution to a non-linear partial differential equation, which cannot be solved explicitly in general. To obtain closed-form solutions, Kim and Omberg (1996) disallow inter-temporal consumption. Wachter (2002) assumes complete markets and allows consumption during life time, but her analysis is restricted to constant relative risk aversion (CRRA). Liu (2007) allows quite general quadratic asset return dynamics and solves the problem under CRRA up to the solution of a system of ordinary differential equations. To the best of our knowledge, our paper is the first to provide a closed-form solution to the portfolio choice problem for a constant absolute risk aversion (CARA) investor with inter-temporal consumption and a finite horizon, facing mean-reverting expected returns. This enables us, for the first time, to analyze both qualitatively as well as quantitatively the horizon effect on a CARA investor's demand for risky asset due to inter-temporal hedging.

2. A general model of time diversification

To set the stage for our analysis of time diversification, we begin with a two-asset version of Merton's (1969, 1971) canonical model of portfolio choice in continuous-time. A consumer can invest in a risk-free asset with the constant rate of return r, and in a risky asset whose rate of return follows the diffusion process:

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[6] Liu and Pan (2003), Branger, Schlag, and Schneider (2008) consider stochastic volatility and jumps in the portfolio choice problem. Normandin and St-Amour (2008), Cardaka and Wilkins (forthcoming) conduct empirical analysis of household portfolios.

$$\frac{dP(t)}{P(t)} = \alpha(t)dt + \sigma_p dz_p(t), \tag{1}$$

where z_p is a standard-normal Wiener process. The expected rate of return α also follows a diffusion process:

$$d\alpha(t) = \mu_\alpha(t)dt + \sigma_\alpha dz_\alpha(t), \tag{2}$$

where z_α is a Wiener process, and may be correlated with the innovation to the rate of return:[7]

$$dz_p dz_\alpha = \rho_{p\alpha} dt, \rho_{p\alpha} >=< 0. \tag{3}$$

Define the proportion of wealth invested in the risky asset at time t as $\lambda(t)$. Assume that the consumer receives wage income of $Y(t)dt$ in each period and, for simplicity, that the flow of wage income is constant, $Y(t) = \bar{Y}, \forall t.$

Given initial wealth W_0, the consumer's wealth accumulates according to the budget constraint:

$$dW = \{[(1-\lambda)r + \lambda\alpha]W + Y - C\}dt + \lambda\sigma_p W dz_p. \tag{4}$$

The consumer lives T periods, has time-separable preferences defined over consumption, and maximizes expected lifetime utility:

$$E_0\left\{\int_0^T U[C(t),t]dt + \Omega[W(T),T]\right\}. \tag{5}$$

The felicity function $U[C,t]$ is increasing and concave in C. The bequest function $\Omega[W,T]$ is increasing and strictly concave in W.

Applying Theorem 5.1 of Merton (1971) *mutatis mutandis*, optimal policies for this problem exist and satisfy the Bellman equation:

$$0 = \max_{C,\lambda} \quad U(C,t) + J_t + J_w\{[(1-\lambda)r + \lambda\alpha]W + Y - C\} + J_{ww}\lambda^2 W^2 \frac{\sigma_p^2}{2} + J_\alpha \mu + J_{\alpha\alpha}\frac{\sigma_\alpha^2}{2} + J_{w\alpha}\lambda W \sigma_{p\alpha} \tag{6}$$

where

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[7] Alternatively, we could use the Sharpe ratio, $(\alpha - r)/\sigma_p$, as the second state variable. See Kim and Omberg (1996).

$$J\left[W(t),t\right]=\max_{\{C,\lambda\}} E_t \int_t^{\infty} U(C,s)ds+\Omega\left[W(T),T\right] \tag{7}$$

is the value function, the subscripts attached to $J\left[W(t),t\right]$ denote partial derivatives, $\sigma_{p\alpha}=\sigma_p\sigma_\alpha\rho_{p\alpha}$ and the time indices have been deleted to simplify notation.

The first-order condition for consumption requires that the marginal utility of consumption is equal to the marginal utility of wealth

$$U_c=J_W. \tag{8}$$

More importantly for our purpose, the first-order condition for λ requires that the demand for the risky asset satisfies

$$\lambda(W,\alpha,t,T)=-\frac{J_W(\alpha-r)}{J_{WW}W\sigma_p^2}-\frac{J_{W\alpha}\sigma_{p\alpha}}{J_{WW}W\sigma_p^2}. \tag{9}$$

The first term on the right hand side of Eq. (9) is the speculative demand for the risky asset, since it is proportional to the expected excess return, $\alpha(t)-r$. The second term, proportional to $\sigma_{p\alpha}$, is the hedging demand, since it is the value of λ that minimizes the variance of the portfolio. We include the terminal period T as a potential argument in the portfolio demand.

We are now equipped to formalize the two alternative notions of time diversification discussed earlier. The first notion is epitomized by Samuelson's (1989, p. 9) statement of " . . . the surprising theorem that your uncle with few years to go should hold exactly the same fraction of wealth in risky (random walk) stocks as he did when young and as you do now."[8] This suggests a cross-sectional interpretation of time diversification. Imagine two people who have the same wealth and face the same expected return from the risky asset at the same time. Time diversification occurs if the younger person invests a larger proportion of his wealth in the risky asset than the older person does. To express this formally, consider two investors, 1 and 2, with the same preferences, who have the same wealth *W(t)*, and who share a common expected return *α(t)* at time t. However, investor 2 has a longer planning horizon than investor 1, so that $T_2>T_1$. We propose

Definition 1: *Cross-sectional diversification occurs if* $\forall t, t < T_1 < T_2,\ \lambda(W(t), \alpha(t), t, T_2) > \lambda(W(t), \alpha(t), t, T_1)$.

Alternatively, this can be interpreted as applying to a single person at two different points in time if he happens to have the same wealth and same expected return at those distinct points in time. The important thing to note is that cross-sectional diversification is a *ceteris paribus* concept that depends on both wealth and the expected return.[9]

Cross-sectional time diversification asserts that a person with a longer planning horizon should invest more in the risky asset than a person with a shorter planning horizon, *ceteris paribus.* This is exactly what Samuelson (1994, p. 17) shows cannot occur in the canonical model of portfolio choice with CRRA utility and random walk returns in Samuelson (1969) and Merton (1969,1971): "Then it is an exact theorem that *The Investment Horizon Can Have No Effect on Your Portfolio Proportion*."[10] Cross-sectional diversification seems to be what Jagannathan and Kocherlakota (1996) have in mind in their title, "Why should older people invest less in stocks than younger people?" Bodie and Crane (1997) and Ackert, Church, and Englis (2002) undertake explicit tests for cross-sectional diversification.[11]

Our second definition draws from the conventional wisdom dispensed by financial advisors. For example, Quinn (1991, p. 489) recommends that investors ". . . tip toward higher risks" when young. Similarly, Malkiel (1996, p. 411)] suggests that ". . . the longer the time period over which you can hold your investments, the greater should be the share of common stocks in your portfolio." This suggests a time-series interpretation of

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[8] These are Samuelson's italics. See Samuelson (1990, p.6).

[9] Without giving it a name, Kim and Omberg (1996) implicitly adopt a similar idea of time diversification: Suppose that the portfolio demand depends upon the planning horizon $h = T - t$ so that we can write $\lambda(W_t, \alpha_t, h)$. They derive conditions where $\partial\lambda/\partial h > 0$. This is essentially a "local" version of our notion of cross-sectional diversification: Imagine a person who has a given level of wealth and expected return. If a small increase in the planning horizon causes the demand for the risky asset to increase, then cross-sectional diversification occurs. This only applies to marginal changes in horizon, while we are interested in the more general "global" definition.

[10] This is again Samuelson's emphasis.

[11] Bodie and Crane (1997) run a cross-sectional regression of the equity share on a range of explanatory variables, of which age, home ownership, and net worth are significant. Ackert, Church, and Englis (2002) investigate statistically how portfolio demands differ in a cross-section of heterogeneous investors, where the investors differ in gender, home ownership, net worth, and psychological orientation, as well as age.

time diversification: as the investor looks into the future, he should plan to hold less of the risky asset as he ages, allowing for the expected accumulation of his wealth over time and the long run behavior of the expected rate of return. Formally, consider an investor at time t who expects to live T years. He has wealth W_t and faces the expected rate of return α_t. He anticipates his demand for the risky asset at time $\tau, t < \tau < T$. We have

Definition 2: *Time-series diversification occurs if* $\forall t, t < \tau < T,\ E_t \lambda(W(\tau), \alpha(\tau), \tau, T) > \lambda(W(t), \alpha(t), t, T)$, *where* E_t *stands for the expectation conditional on the information available at time t.*

Notice that, unlike cross-sectional diversification, time-series diversification is not a *ceteris paribus* concept. It holds neither wealth, nor the expected return, nor even the planning horizon constant. It is merely a statement about how an investor might expect his portfolio to change as he ages, allowing for these things to change. In fact, it has nothing intrinsically to do with age, or planning horizon *per se*. An investor with an infinite planning horizon could still find it desirable to practice time-series diversification. Similarly, two investors with different ages could plan to change their portfolios in the same way. What matters for time-series diversification is the forecast horizon, not the planning horizon.

It is important to note that these two concepts are often confounded. Different authors use the term “time diversification” in different ways, causing needless confusion. We suspect, for example, that time-series diversification is much closer to what financial advisors have in mind than cross-sectional diversification. A rational investor would surely take expected changes in his wealth and expected returns into consideration in planning his lifetime portfolio strategy. However, academic researchers–in refuting time diversification–often seem to have cross-sectional diversification in mind. One academic paper of which we are aware that uses a time-series notion of time diversification is the empirical paper by Levy and Spector (1996). Following Samuelson’s example, Levy and Spector adopt CRRA utility. Using historical data from 1926 to 1990 they calculate optimal portfolios for the entire sample period and shorter sub-periods. They find that although equities dominate bonds over the entire period, the optimal portfolio holds a smaller proportion of equity over shorter horizons. We will show that this can easily be explained as time-series diversification resulting from changes in the expected return, changes which will differ depending upon the length of the forecast horizon.

The same authors may confound the two notions even in the same paper. As we have seen, the very title of Jagannathan and Kocherlakota's (1996) paper, "Why should older people invest less in stocks than younger people?" evokes cross-sectional diversification. However, their introduction (p. 11) cites the conventional wisdom that ". . . investors should switch from stocks to bonds as they age . . .," which implies time-series diversification. Nonetheless, their basic mathematical model leads to the proposition (p. 16) that ". . . long investment horizons are no different from short investment horizons as long as households are making decisions at regular intervals." This sounds like cross-sectional diversification. Later, however, they look at how a portfolio evolves over time (p. 18), which is a time-series question.

3. Time diversification with HARA preferences

At the level of generality of the portfolio demand in Eq. (9), it is not clear how either kind of time diversification can occur. To say more, we must impose a bit more structure by assuming a particular class of preferences. Suppose that the investor exhibits hyperbolic absolute risk aversion (HARA) preferences. The felicity function in Eq. (5) then takes the form

$$U(C) = \frac{1-\gamma}{\gamma}\left(\frac{\beta C}{1-\gamma}+\eta\right)^{\gamma}, \tag{10}$$

where, following Merton (1971),

$$\gamma \neq 1, \beta > 0, \frac{\beta C}{1-\gamma}+\eta > 0, \text{ and } \eta = 1 \text{ if } \gamma \to -\infty. \tag{11}$$

As noted by Merton (1971), this class of preferences subsumes all of the standard, time-separable preferences frequently employed in the economics and finance literature [quadratic $(\gamma = 2)$, constant relative risk aversion $(\eta = 0)$, and constant absolute risk aversion $(\gamma \to -\infty)$]. If $\eta > 0$, the consumer enjoys a kind of guaranteed minimal consumption. Unlike Merton (1971), we allow the possibility that $\eta < 0$, which captures the notion of subsistence income (Samuelson, 1994) or target consumption (Jagannathan and Kocherlakota, 1996).

As shown by Merton [1971, p. 402, Eq. (98)], the value function in this case is also of the HARA form:[12]

$$J(W,\alpha,t)=\frac{1-\gamma}{\gamma}e^{-\rho t}H(\alpha,t)\left\{\frac{W}{1-\gamma}+\frac{\eta+Y}{\beta r}\left(1-e^{-r(t-T)}\right)\right\}^{\gamma}+L(\alpha,t). \tag{12}$$

The functions $H(\alpha_t,t,T)$ and $L(\alpha_t,t,T)$ are solutions to the partial differential equations:

$$0=\left(\frac{H}{\beta}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{\beta-\gamma}{\beta\gamma}\right)-H\frac{\rho}{\gamma}+H_t\frac{1}{\gamma}+Hr+\frac{\left[H(\alpha-r)+H_\alpha\sigma_{\alpha p}\right]^2}{2H(1-\gamma)\sigma_p^2}+H_\alpha\frac{\mu}{\gamma}+H_{\alpha\alpha}\frac{\sigma_\alpha^2}{2\gamma}, \tag{13}$$

$$0=L_t+L_\alpha\mu+L_{\alpha\alpha}\frac{\sigma_\alpha^2}{2}, \tag{14}$$

subject to the boundary condition

$$J(W_T,\alpha_T,T)=\Omega(W_T,\alpha_T,T). \tag{15}$$

Because of the separability of the value function, the portfolio demand in Eq. (9) reduces to

$$\lambda=\left[\frac{\alpha-r}{\sigma_p^2}+\frac{H_\alpha(\alpha,t,T)}{H(\alpha,t,T)}\frac{\sigma_{p\alpha}}{\sigma_p^2}\right]\left\{\frac{1}{1-\gamma}+\frac{\eta+Y}{\beta rW}\left(1-e^{r(t-T)}\right)\right\}. \tag{16}$$

The expression $(\eta+Y)\left(1-e^{r(t-T)}\right)$ is the discounted present value of wage income plus "guaranteed" consumption. Eq. (16) gives us some clues about when cross-sectional and time-series diversification may or may not occur.

We first consider cross-sectional time diversification. We are comparing two individuals of different ages (different $T's$), at the same time *t*, the same wealth *W(t),* the same wage income Y, and the same expected rate of return. Inspection of Eq. (16) leads to the following proposition:

Proposition 1: *Cross-sectional diversification can occur only under three circumstances: (1)* $Y>0$, *(2)* $\eta>0$, *or (3) the function* H_α/H *depends upon the difference in the ages of the two individuals.*

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[12] Eq. (12) differs slightly from Merton's Eq. (98) because we allow for non-asset income and use a slightly different price process. Because of these minor differences, we include the derivation of Eq. (12) in Appendix A.

The first case is well established (Samuelson, 1994; Jagannathan and Kocherlakota, 1996): risk-free wage income is equivalent to the income generated from a risk-free annuity. The older individual expects to receive this risk-free flow of income for fewer years than the younger individual, so he will invest a larger share of his financial wealth in the risk-free asset. Formally, consider individual 1, who will die at T_1 and individual 2, who will die at T_2, with $T_2 > T_1$. The share of financial wealth invested in the risky asset at time t by individual 1 (the older) will be smaller than that of individual 2 (the younger) if $Y > 0,$ *ceteris paribus.*[13]

The second case is essentially the same because the guaranteed consumption $(\eta > 0)$ has the same effect as an increase in permanent income. Notice, however, that–as noted by Samuelson (1994) and Jagannathan and Kocherlakota (1996)–subsistence or target consumption $(\eta < 0)$ has exactly the opposite effect. Target consumption tends to make older people hold riskier portfolios than younger people.

In the third case, cross-sectional diversification occurs when the hedging demand is a function of the planning horizon. We provide an example of when it can occur in the next section.

Now consider time diversification. Again glancing at Eq. (16), we have the following proposition:

Proposition 2*: Time-series diversification can occur under three circumstances: (1)* $Y > 0$ *or* $\eta > 0$, (2) *the function* H_α / H *is expected to change over time, or (3)* α *is expected to decline over time.*

Cases (1) and (2) are essentially the same as with cross-sectional diversification, with one caveat. Cross-sectional diversification requires human wealth or the hedging demand to depend upon the planning horizon. Time-series diversification requires human wealth or the anticipated hedging demand to decrease over time, regardless of the planning horizon. In other words, time-series diversification revolves around the length of the forecast horizon, rather than the planning horizon. An investor with an infinite horizon would find time-series diversification as beneficial as an investor with a finite horizon, provided they were projecting their portfolio demands the same length of time into the future. Both would have the same answer to the question, “How much less of the risky asset should I expect to hold five years from now?” Case (3) asserts that a sufficient condition

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[13] Bodie, Samuelson, and Merton (1992) expand upon this reasoning by endogenizing labor supply. A younger person should

for time-series diversification to occur is simply that the investor anticipates that the expected rate of return to fall over time.

4. Two cases with closed-form solutions

We will take the case for time diversification based upon human wealth as well established.[14] Henceforth, we will abstract from this case by setting Y and η to zero. The model can then be interpreted as modeling wage income as riskless, whose present value is included as part of the initial total wealth. This will allow us to focus on two new possibilities: first, that both cross-sectional and time-series diversification may occur through the hedging component of the portfolio; second, that anticipated changes in the expected rate of return might drive time-series diversification.

The propositions in the preceding section suggest these possibilities might occur, but do not establish that they must occur. To show that they can, we demonstrate two cases with closed-form solutions. The first is a case of pure time diversification. The second allows both forms of time diversification to occur simultaneously.

4.1. Pure time-series diversification

Consider the canonical model considered by Samuelson (1969) and Merton (1969, 1971). Investors exhibit constant relative risk aversion (CRRA): this is a special case of the preferences in Eq. (9) where $\eta = 0$ and $\beta = 1$. We also consider a special case of Eq. (2) by letting $\mu_{\alpha}(t) = \theta[\mu - \alpha(t)]$ and assuming that the instantaneous standard deviation $\sigma_{\alpha} = \sigma$ is a constant. In this case the expected rate of return follows an Ornstein-Uhlenbeck process: $\mu > 0$ is the steady-state mean of $\alpha(t)$ and $\theta > 0$ governs the speed of convergence to the steady state. In order to abstract from hedging effects, we assume that the innovation to $d\alpha(t)$ is uncorrelated with the innovation with the rate of return, so $\sigma_{p\alpha} = 0$. From Eq. (16) we arrive at the familiar portfolio demand

$$\lambda(t) = \frac{\alpha(t) - r}{(1-\gamma)\sigma_p^2}. \tag{17}$$

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invest more in risky assets than an older person because he can work more to recoup his losses in stocks.

[14] See for example, Koo (1999), Heaton and Lucas (1997), Viceira (2001), Benzoni et al. (2007), and Polkovnichenko (2007).

The portfolio demand is independent of the planning horizon, as shown by Samuelson (1969) and Merton (1969, 1971). However, this only proves that cross-sectional diversification cannot occur. What about time-series diversification? Imagine an investor standing at time t trying to predict what his portfolio demand will be at some time τ in the future, $t < \tau < T$. It is obvious that the expected share of risky asset in his portfolio will be

$$E_t \lambda(\tau) = \frac{E[\alpha(\tau) \mid \alpha(t)] - r}{(1-\gamma)\sigma_p^2}. \tag{18}$$

Since α is distributed normally, it is possible to calculate this expectation explicitly:

$$E_t \lambda(\tau) = \frac{\mu + (\alpha(t) - \mu)e^{-\theta(\tau - t)} - r}{(1-\gamma)\sigma_p^2}. \tag{19}$$

Now consider the expected change in the portfolio between periods t and τ, conditional upon the expected return at time t, $\alpha(t)$:

$$E_t \lambda(\tau) - \lambda_t = \frac{(\mu - \alpha(t))(1 - e^{-\theta(\tau - t)})}{(1-\gamma)\sigma_p^2}. \tag{20}$$

Time-series diversification requires that this to be negative. Clearly, a necessary and sufficient condition for time-series diversification to obtain is that the current expected rate of return be always greater than its long run mean μ. When the innovation to $d\alpha$ is uncorrelated with the innovation to rate of return, there is no hedging demand. Allocation to the risky asset is linear in the expected return. If the current expected rate of return $\alpha(t)$ is higher than its long run mean and $\alpha(t)$ is expected to decline over time due to mean-reversion, the demand for risky asset is expected to fall with it. This result is consistent with the prediction of Proposition 2.

4.2. *Cross-sectional diversification*

Now consider the case when investors have constant absolute risk aversion (CARA) utility: this a special case of the preferences in Eq. (9) where $\gamma \to \infty$. Assume that the bequest function is also CARA:

$$\Omega[W(T), T] = -e^{-\omega W_T}. \tag{21}$$

As in Merton (1971, p. 407, Eq.(120)), the price dynamics in Eq. (1) take a special form: the instantaneous standard deviation of the rate of price growth is constant, so $\sigma_p = \sigma$; however, the expected rate of return evolves according to a special case of Eq. (2):

$$\begin{aligned} d\alpha = {} & \theta(\mu-\alpha)dt + \delta\left(\frac{dP}{P} - \alpha dt\right) \\ = {} & \theta(\mu-\alpha)dt + \delta\sigma dz. \end{aligned} \tag{22}$$

As explained by Merton, Eq. (22) asserts that the change in the expected return has two components: a long-run regressive adjustment towards the steady-state mean, $\mu > 0$ and a short run "error-learning" (adaptive expectations) adjustment. Notice that α is an Ornstein-Uhlenbeck process.

Under these conditions, it can be shown that the portfolio share invested in the risky asset is (see Appendix B):

$$\lambda = \frac{\alpha - r}{\eta r\sigma^2 W} - \frac{\delta}{\eta r\sigma^2 W}\left\{\frac{1}{x_2}\left[\frac{\theta\mu + \delta r}{x_1} - r\right] + \frac{\theta\mu + \delta r}{x_1(\delta+\theta)}e^{x_1(t-T)} - \frac{\theta(\mu - r)}{x_2(\delta+\theta)}e^{x_2(t-T)} + \frac{\alpha}{x_1}\left(1 - e^{x_1(t-T)}\right)\right\} \tag{23}$$

where $x_1 = r + 2(\delta+\theta)$ and $x_2 = r + \delta + \theta$. For the infinite horizon case $(T \to \infty)$, Eq. (23) reduces to the solution in Merton [1971, p. 408, Eq. (129)]:

$$\lambda = \frac{\alpha - r}{\eta r\sigma^2 W} - \frac{\delta}{\eta r\sigma^2 W}\left\{\frac{1}{x_2}\left[\frac{\theta\mu + \delta r}{x_1} - r\right]\right\}. \tag{24}$$

We now consider cross-sectional time diversification. Imagine two investors in the same time period t who face the same expected return $\alpha(t)$. They are identical in all respects except that one is younger than the other. Denote the planning horizon of the younger investor by $h_y = T_y - t$ and the planning horizon of the older investor by $h_o = T_o - t$, where $T_o < T_y$ so that $h_y > h_o$. When will $\lambda(\alpha(t), h_y) > \lambda(\alpha(t), h_o)$? To address this question we derive in Appendix C a set of graphs that show how the demand for the risky asset is affected by the planning horizon, conditional upon a given $\alpha(t)$. Cross-sectional diversification occurs when this curve is positively sloped, so that demand for the risky asset increases with the length of the planning horizon.

To produce quantitative results, we calibrate the model using market parameters calculated based on those in Wachter (2002). In particular, we let $\mu = 0.0048$, $r = 0.0014$, $\sigma = 0.0436$, $\theta = 0.0226$, and $\delta = \pm 0.0189$. All parameters are in monthly units. We choose this set of parameters because they are estimated from market data in previous studies and thus reflect asset dynamics in the real world (see Appendix D for the calibration method).

First, we examine the case when next period's expected return is positively related to this period's realized return, i.e., $\delta > 0$ (although this is not the interesting case empirically). Define $\Gamma = \frac{\theta\mu + \delta r}{\delta + \theta}$. There are three possible scenarios, depending on the magnitude of α_t (for illustrative purposes, we look at the cases when $\alpha = \Gamma + 0.0003$, $\alpha = \mu$, and $\alpha = r - 0.0003$ respectively. These choices are both empirically plausible and are representative of three different scenarios.):

1a. If $\alpha(t) > \Gamma$ then λ is globally decreasing and convex in the horizon *h*, tending asymptotically to the infinite horizon demand in Merton, λ_m, as the planning horizon goes to infinity $(h \to \infty)$. This is depicted in Figure 1.

2a. If $\Gamma > \alpha(t) > r$ then λ is still globally decreasing in h, but may initially be convex or concave. Figure 2 depicts the case where it is initially convex.

3a. If $r > \alpha(t)$ then λ is initially *increasing* in h, reaches a maximum, and then falls to λ_m. This is depicted in Figure 3.

This yields the following proposition:

***Proposition 3**: When next period's expected return is positively related to this period's realized return, i.e., $\delta > 0$, a young investor holds less of the risky asset than an older investor if the expected rate of return is higher than the risk-free rate $(\alpha(t) > r, \forall t)$. The opposite occurs only at very short horizons and if the expected rate of return is sufficiently low, so that $(\alpha(t) < r, \forall t)$. However, the magnitude of the horizon effect on portfolio demand is very small.* (Proof is given in Appendix C under "discussions".)

In other words, when $\delta > 0$, the opposite of cross-sectional diversification occurs, i.e., older investors could hold *more* of the risky asset than younger investors if the expected rate of return is reasonably high.

From Eq. (23), we can decompose the portfolio demand for risky asset into three parts:

$$\begin{aligned}\lambda W &= \frac{\alpha(t)-r}{\eta r\sigma^2} \\ &-\frac{\delta}{\eta r\sigma^2 x_1}\left[\frac{\theta(\mu-r)}{x_2}+(\alpha(t)-r)\right] \\ &+\frac{\delta}{\eta r\sigma^2(\delta+\theta)}\left\{\frac{\theta(\mu-r)}{x_2}e^{-x_2 h}+\frac{\delta(\alpha(t)-r)}{x_1}e^{-x_1 h}+\frac{\theta(\alpha(t)-\mu)}{x_1}e^{-x_1 h}\right\}\end{aligned} \tag{25}$$

The first part denotes the myopic demand. The second part represents the hedging demand for an infinite horizon investor as that in Merton [1971, p. 408, Eq. (129)]. The first part and second part do not depend on h. The third part does depend on h. Therefore, the third part can be seen as containing finite horizon correction terms. The myopic demand only responds to the current risk premium. In our finite horizon case, such hedging demand is further modified by the correction terms in the third part.

A close examination of the three terms inside the bracket is in order. Because the investor has a finite horizon, he must tradeoff between the overall normal expected stock return and the risk free rate $(\mu - r)$, the current expected stock return and the risk free rate $(\alpha(t) - r)$, and the current expected stock return and the normal expected stock return $(\alpha(t) - \mu)$. The first correction term is positive by the assumption that $\mu > r$. As the horizon h decreases, the first term increases. The sign of the sum of the second term and the third term depends on whether $\alpha(t) >=< \Gamma = \frac{\theta\mu + \delta r}{\delta + \theta}$. If $\alpha(t) > \Gamma$, then the sum is positive. As the horizon h decreases, the demand for stock increases. If $r < \alpha(t) < \Gamma$, then the sum is negative but still dominated by the first term. Therefore, the total demand remains increasing as the horizon h shrinks. Finally, if $\alpha(t) < r$, the sum of the three terms may revert signs as the horizon h decreases, resulting in the portfolio demand first increasing then decreasing as the horizon h decreases.

Next, we examine the case when next period's expected return is positively related to this period's realized return, i.e., $\delta < 0$. In fact, this seems more plausible empirically, according to Pastor and Stambaugh (2006). Again, there are three possible scenarios, depending on the value of $\alpha(t)$:

1b. If $\alpha(t) > \Gamma, \forall t$, then λ increases at a decreasing rate as h increases, tending asymptotically to the demand in Merton, λ_m, as the planning horizon goes to infinity $(h \to \infty)$. This is depicted in Figure 4. Note that with the set of parameters we have chosen, $\alpha(t) > \Gamma = 26.6\%$ per annum, making case 1b an unlikely case empirically.

2b. If $\Gamma > \alpha(t) > r, \forall t$, then λ is still globally increasing in h, but may initially be convex or concave, depending upon whether the inflection point occurs at a negative or positive value of h. Figure 5 depicts the case where it is initially convex. This set of parameters makes case 2b the most empirically relevant scenario.

3b. If $\alpha(t) < r, \forall t$, then λ is initially *decreasing* in h, reaches a minimum, and then increases to λ_m. This is depicted in Figure 6.

The following proposition summarizes the above discussion:

Proposition 4*: When next period's expected return is negatively related to this period's realized return ($\delta < 0$), a young investor holds more of the risky asset than an older investor if the expected rate of return is higher than the risk-free rate $(\alpha(t) > r, \forall t)$. The magnitude of the horizon effect on portfolio demand is large.*

In other words, if next period's expected return is negatively related to this period's realized return ($\delta < 0$), time diversification holds uniformly almost all the time except when the expected return is extremely low and at very short horizons.

Overall, Figure 1 through Figure 6 indicate that the quantitative results for portfolio demand seem more realistic empirically when $\delta < 0$ (the case implied by data) than those obtained when $\delta > 0$. Moreover, when $\delta < 0$, the portfolio demand varies significantly across agents who are otherwise identical except with different horizons. When $\delta = 0$, hedging demand is zero and does not induce horizon effect.

There is an interesting similarity between the conditions for time diversification in Proposition 3 and Proposition 4 that involve inequalities like $\alpha(t) > r, \forall t$ and the conditions that imply bounds on the expected returns of contingent claims [See Propositions 5 and 6 in Grundy (1991), Lo (1987), de la Pena et al. (2004).] Similar conditions arise in several problems in financial economics besides those considered in this paper.

5. Conclusion

We establish general conditions under which younger investors should invest a larger proportion of his wealth in risky assets than older investors, a phenomenon known as time diversification. Specifically, we clarify two different notions of time diversification often confounded in the literature. Cross-sectional time diversification refers to the situation where a younger person invests a larger share of his wealth in risky asset than an older person with the same amount of wealth and faces identical expected stock return. Time-series diversification describes the case where the same person is expected to reduce his share invested in risky asset as he ages. In addition to the well-established results with human wealth and target consumption, we show that time diversification can be caused by investor's hedging demand if the change in the expected rate of return is correlated with the current rate of return.

Using a case with closed-form results, we illustrate that a CRRA investor expects to reduce his share of wealth in risky asset if the current expected return is higher than its long run mean and is expected to decline, leading to time-series diversification. In the second case, we, for the first time, provide a close-form solution to the portfolio choice problem for a CARA investor with inter-temporal consumption and a finite horizon, facing mean-reverting expected returns. We show that cross-sectional time diversification can obtain if next period's expected return is negatively correlated to this period's realized return as long as the current expected return is higher than the risk-free rate. Moreover, a CARA investor's horizon can affect the size of the portfolio demand significantly due to inter-temporal hedging.

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Appendix A

Derivation of the portfolio demand for HARA preferences

The Bellman equation is now

$$
\begin{aligned}
0 = \max_{C,\lambda} \quad & e^{-\rho t}\frac{1-\gamma}{\gamma}\left(\frac{\beta C}{1-\gamma}+\eta\right)^{\gamma} + J_t + J_w\{[(1-\lambda)r+\lambda\alpha]W+Y-C\} + J_{ww}\lambda^2 W^2\frac{\sigma_p^2}{2} \\
& + J_\alpha \mu + J_{\alpha\alpha}\frac{\sigma_\alpha^2}{2} + J_{\alpha w}\lambda W\sigma_{p\alpha}.
\end{aligned}
\tag{A.1}
$$

Maximizing over C and λ yields the first-order conditions

$$\beta e^{-\rho t}\left(\frac{\beta C}{1-\gamma}+\eta\right)^{\gamma-1} = J_w, \tag{A.2}$$

$$J_w(\alpha-r)W + J_{ww}\lambda W^2\sigma_p^2 + J_{\alpha w}W\sigma_{p\alpha} = 0. \tag{A.3}$$

Solving (A.3) yields the portfolio demand Eq. (9) in the text. To determine the value function, substitute (A.2) and (A.3) back into the Bellman Eq. (A.1) to arrive at the following partial differential equation:

$$
\begin{aligned}
0 = {} & (1-\gamma)e^{-\rho t}\left(\frac{J_w}{\beta}e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{\beta-\gamma}{\beta\gamma}\right) + J_t + J_w\left[rW+Y+(1-\gamma)\frac{\eta}{\beta}\right] - \frac{\left[J_w(\alpha-r)+J_{\alpha w}\sigma_{\alpha p}\right]^2}{2J_{ww}\sigma_p^2} \\
& + J_\alpha\mu + J_{\alpha\alpha}\frac{\sigma_\alpha^2}{2}.
\end{aligned}
\tag{A.4}
$$

Using the conjectured solution in Eq. (12) of the text, (A.4) reduces to the pair of partial differential equations in Eq. (13) and (14).

Appendix B

Derivation of the portfolio demand with CARA preferences

Given CARA preferences (γ → -∞), and the Ornstein-Uhlenbeck process for α in Eq. (22) of the text, the Bellman equation becomes

$$
\begin{aligned}
0 = \max_{\lambda,C} \quad & -\frac{e^{-\rho t-\eta C}}{\eta} + J_t + J_w\{[(1-\lambda)r+\lambda\alpha]W-C\} + J_{ww}\lambda^2 W^2\frac{\sigma^2}{2} + \\
& J_\alpha\theta(\mu-\alpha) + J_{\alpha\alpha}\delta^2\frac{\sigma^2}{2} + J_{w\alpha}\lambda W\delta\frac{\sigma^2}{2}.
\end{aligned}
\tag{B.1}
$$

The first-order condition for *C* in Eq. (A.2) of Appendix A becomes

$$e^{-\rho t-\eta C}=J_w. \tag{B.2}$$

The first-order condition for λ is the same as (A.3) in Appendix A, except that $\sigma_{p\alpha}=\delta\sigma^2$.

Substituting the first-order condition back into the Bellman equation yields the partial differential equation

$$0=-\frac{J_w}{\eta}+J_t+J_w rW-\frac{\left[J_w(\alpha-r)+J_{w\alpha}\delta\sigma^2\right]^2}{2J_{ww}\sigma^2}+\frac{J_w}{\eta}(\rho t+\ln J_w)+J_\alpha\theta(\mu-\alpha)+J_{\alpha\alpha}\delta^2\frac{\sigma^2}{2}. \tag{B.3}$$

We seek a value function $J(W,\alpha,t)$ that satisfies (B.3) and the boundary condition

$$J(W_T,\alpha_T,T)=-e^{-\omega W_T}. \tag{B.4}$$

Conjecture that the value function is of the form

$$J(W,\alpha,t)=-e^{-a(t)-B(t)W-D(t)\alpha-F(t)\frac{\alpha^2}{2}}, \tag{B.5}$$

where *A(t), B(t), D(t), and F(t)* are unknown functions of time. Using this guess, (B.3) becomes

$$\begin{aligned}0=&-\frac{B}{\eta}+A'+B'W+D'\alpha+F'\frac{\alpha^2}{2}+BrW+\frac{\left[\alpha-r-(D+F\alpha)\delta\sigma^2\right]^2}{2\sigma^2}+\\&\frac{B}{\eta}\left[\rho t+\ln B-A-BW-D\alpha-F\frac{\alpha^2}{2}\right]+\theta(D+F\alpha)(\mu-\alpha)+\left[F-(D+F\alpha)^2\right]\delta^2\frac{\sigma^2}{2}.\end{aligned} \tag{B.6}$$

Collecting terms, this can be expressed as

$$\begin{aligned}0=&\left\{A'-B\frac{A}{\eta}-\frac{B}{\eta}+\frac{r^2}{2\sigma^2}+\frac{B}{\eta}(\rho t+\ln B)+(\theta\mu+\delta r)D+F\delta^2\frac{\sigma^2}{2}\right\}+\left\{B'+rB-\frac{B^2}{\eta}\right\}W+\\&\left\{D'-\left(\frac{B}{\eta}+\theta+\delta\right)-\frac{r}{\sigma^2}+(\theta\mu+\delta r)F\right\}\alpha+\left\{F'-\left[\frac{B}{\eta}+2(\theta+\delta)\right]F+\frac{1}{\sigma^2}\right\}\frac{\alpha^2}{2}.\end{aligned} \tag{B.7}$$

Since this must hold for all $W,\alpha,$ and $t,$ Eq. (B.7) reduces to a system of four ordinary differential equations in time. Using the boundary condition (B.4) in conjunction with the conjectured value function (B.5), there are four corresponding boundary conditions: $A(T)=0,$ $B(T)=\omega,$ $D(T)=0,$ and $F(T)=0.$ This system of boundary value problems may be solved recursively, as follows.

First, consider the function B (t), which satisfies the differential equation

$$0 = B' + rB - \frac{B^2}{\eta}. \tag{B.8}$$

This is a logistic equation with the general solution

$$B(t) = \frac{\eta r}{1 + \eta r k_1 e^{rt}}, \tag{B.9}$$

where k_1 is an arbitrary constant. Using the boundary condition B (T) = ω, we can peg down the constant:

$$k_1 = \frac{\eta r - \omega}{\eta r} e^{-rT}. \tag{B.10}$$

Thus, we arrive at

$$B(t) = \frac{\eta r}{1 + (\eta r - \omega) e^{r(t-T)}}. \tag{B.11}$$

Notice that in the infinite horizon limit, as $T \to \infty$, $B(t) \to \eta r$, the familiar coefficient attached to wealth in the value function in Merton's (1969,1971) model with CARA preferences. Notice also that if ω= ηr, then B (t) = ηr for all t. To simplify exposition we will focus on this case.

Next, consider the function $F(t)$. Assuming that $B = \eta r$, $F(t)$ satisfies the linear, first-order differential equation:

$$F' - F[r + 2(\delta + \theta)] = -\frac{1}{\sigma^2}. \tag{B.12}$$

Using the boundary condition $F(T) = 0$, it is easy to show that

$$F(t) = \frac{1}{\sigma^2 x_1}\left[1 - e^{x_1(t-T)}\right], \tag{B.13}$$

where $x_1 = r + 2(\delta + \theta)$.

Now consider the function $D(t)$. Again assuming that $B = \eta r$, $D(t)$ satisfies

$$D' - D(r + \delta + \theta) = \frac{r}{\sigma^2} - F(\theta\mu + \delta r). \tag{B.14}$$

Substituting (B.13) for $F(t)$ and using the boundary condition $D(T)=0$ yields

$$D(t)=\frac{1}{x_2\sigma^2}\left[\frac{\theta\mu+\delta r}{x_1}-r\right]+\frac{\theta\mu+\delta r}{x_1(x_1-x_2)\sigma^2}e^{x_1(t-T)}-\frac{\theta(\mu-r)}{x_2(x_1-x_2)\sigma^2}e^{x_2(t-T)}, \tag{B.15}$$

where $x_2=r+\delta+\theta.$

Finally, return to the function $A(t)$. When $B=\eta r$, $A(t)$ is determined by

$$A'-rA=r-\frac{r^2}{2\sigma^2}-r(\rho t+\ln\eta r)-D(\theta\mu+\sigma r)-F\delta^2\frac{\sigma^2}{2}. \tag{B.16}$$

Substituting (B.13) and (B.15) for $D(t)$and $F(t)$ into (B.16) and using the boundary condition $A(T)=0,$ we find that

$$A(t)=A_0\left[1-e^{r(t-T)}\right]+\rho\left[t-Te^{r(t-T)}\right]+A_1\left[e^{x_1(t-T)}-e^{r(t-T)}\right]+A_2\left[e^{x_2(t-T)}-e^{r(t-T)}\right], \tag{B.17}$$

where

$$A_0=\rho-r+r\ln r\rho+\frac{\theta\mu+\delta r}{x_1\sigma^2}\left(\frac{\theta\mu+\delta r}{x_1}-r\right)+\frac{\delta^2\sigma^2}{2x_1}, \tag{B.18}$$

$$A_1=\frac{1}{x_1(x_1-r)}\left[\frac{\delta^2\sigma^2}{2}-\frac{(\theta\mu+\delta r)^2}{(\delta+\theta)\sigma^2}\right], \text{ and} \tag{B.19}$$

$$A_2=-\frac{\theta(\theta\mu+\delta r)(\mu-r)}{x_2(\delta+\theta)(x_2-r)\sigma^2}. \tag{B.20}$$

We are now equipped to characterize the optimal consumption and portfolio policies. Using the first-order condition in Eq. (B.3). the consumption function is

$$C(W,\alpha,t)=\frac{A(t)-\rho}{\eta}+\frac{D(t)}{\eta}\alpha+\frac{F(t)}{\eta}\frac{\sigma^2}{2}+rW. \tag{B.21}$$

Using the portfolio condition in Eq. (A.3), the demand for the risky asset is

$$\lambda(\alpha,t)W=\frac{\alpha-r}{\eta r\sigma^2}-\delta\frac{D(t)+F(t)\alpha}{\eta r}. \tag{B.22}$$

Appendix C

Qualitative analysis of cross-sectional diversification in the CARA case

Because the wealth level W is identical for both investors, without loss of generality, we can simplify our discussion by focusing on $X = \lambda W$. Let $h = T - t$ and $\psi = \frac{1}{\eta r \sigma^2}$. Define $\mathrm{A} = \frac{1}{x_2}\left(\frac{\theta\mu + \delta r}{x_1} - r\right)$, $\Gamma = \frac{\theta\mu + \delta r}{\delta + \theta}$, and $\Omega = \frac{\theta(\mu - r)}{\delta + \theta}$. We assume $\mu > r$, so that $\Omega > 0$. It is easy to verify that $\Gamma - \Omega = r$. Denote the total demand for the risky asset by $X(\alpha, h)$. Then Eq. (23) in the text can be written as

$$X(\alpha,h) = \psi\left\{\alpha - r - \delta\left[\mathrm{A} + \frac{\Gamma}{x_1}e^{-x_1 h} - \frac{\Omega}{x_2}e^{-x_2 h} + \frac{\alpha}{x_1}\left(1 - e^{-x_1 h}\right)\right]\right\}. \tag{C.1}$$

Discussion

We only present the case when $\delta > 0$. The case for $\delta < 0$ is similar, just with opposite signs.

1. When $h = 0$ the portfolio demand is (for a given α),

$$X(\alpha,0) = \psi\left\{\alpha - r - \delta\left[\mathrm{A} + \frac{\Gamma}{x_1} - \frac{\Omega}{x_2}\right]\right\}. \tag{C.2}$$

We will assume for convenience that $X(\alpha,0) > 0$, so that the consumer invests some of his terminal wealth in the risky asset (if α < r, X may be less than zero because the process for α does not prevent α to go below r). See footnote 11 in the text.

2. The first derivative of X with respect to h is

$$\frac{\partial X(\alpha,h)}{\partial h} = \psi\delta\left[(\Gamma - \alpha)e^{-x_1 h} - \Omega e^{-x_2 h}\right]. \tag{C.3}$$

Notice that if $\Gamma < \alpha$ then $\partial X(\alpha,h)/\partial h < 0$, so that X is globally decreasing in h. If $\Gamma > \alpha$, however, then X reaches a unique extreme at $\hat{h}$ such that $\partial X(\alpha,\hat{h})/\partial h = 0$. Using Eq. (C.3), and recalling that $x_1 = r + 2(\delta + \theta)$ and $x_2 = r + \delta + \theta$, it follows that

$$\hat{h} = \frac{\ln(\Gamma - \alpha) - \ln \Omega}{\delta + \theta}. \tag{C.4}$$

Note also that

$$\frac{\partial X(\alpha, 0)}{\partial h} = \psi\delta(r - \alpha). \tag{C.5}$$

Therefore $\partial X(\alpha,0)/\partial h >=< 0$ as $r >=< \alpha$. This determines the slope of X at the origin, when $h = 0$.

3. The second derivative of X with respect to h is

$$\frac{\partial^2 X(\alpha,h)}{\partial h^2} = -\psi\delta\left[x_1(\Gamma - \alpha)e^{-x_1 h} - x_2 \Omega e^{-x_2 h}\right]. \tag{C.6}$$

If $\alpha > \Gamma$ then $\partial^2 X(\alpha,h)/\partial h^2 > 0$: X is a globally convex function of h. If $\Gamma > \alpha$ then X has a unique inflection point at $\bar{h}$ such that $\partial^2 X(\alpha,\bar{h})/\partial h^2 = 0$. Using Eq. (C.6) it is straightforward to calculate

$$\bar{h} = \frac{\ln x_1 - \ln x_2 + \ln(\Gamma - \alpha) - \ln\Omega}{\delta + \theta} = \hat{h} + \frac{\ln x_1 - \ln x_2}{\delta + \theta} \tag{C.7}$$

Note that $\bar{h} > \hat{h}$. Furthermore

$$\begin{aligned} \frac{\partial^2 X(\alpha,\hat{h})}{\partial h^2} &= -\psi\delta\left[x_1(\Gamma - \alpha)e^{-x_1 \hat{h}} - x_2 \Omega e^{-x_2 \hat{h}}\right] \\ &= -\psi\delta(\delta + \theta)\Omega e^{-x_2 \hat{h}} < 0. \end{aligned} \tag{C.8}$$

Therefore X reaches a maximum at $\hat{h}$.

4. Finally, in the infinite horizon limit we recover Merton's (1971) solution, which we denote by $X_M(\alpha)$:

$$\lim_{h \to \infty} X(\alpha,h) = \psi\left[\alpha - r - \delta\left(\mathrm{A} + \frac{\alpha}{x_1}\right)\right] = X_M(\alpha). \tag{C.9}$$

Appendix D

Calibration method of the CARA model

Since $\eta = \frac{\gamma}{C}$, where γ is the coefficient of relative risk aversion, Eq. (23) can be re-written using the wealth-to-consumption ratio:

$$\lambda = \frac{1}{\gamma \frac{W}{C}} \left(\begin{array}{l} \frac{\alpha - r}{r\sigma^2} - \frac{\delta}{r\sigma^2 x_1}\left[\frac{\theta(\mu - r)}{x_2} + (\alpha - r)\right] + \\ \frac{\delta}{r\sigma^2(\delta+\theta)}\left\{\frac{\theta(\mu - r)}{x_2}e^{-x_2 h} + \frac{\delta(\alpha - r)}{x_1}e^{-x_1 h} + \frac{\theta(\alpha - \mu)}{x_1}e^{-x_1 h}\right\} \end{array} \right).$$

We then follow Lustig et.al. (2008) to set the wealth-to-consumption ratio[15] to its average value of 87 and Wachter (2002) to calibrate the model with the coefficient of relative risk aversion $\gamma = 10$.

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[15] For monthly data, this ratio becomes 1044.

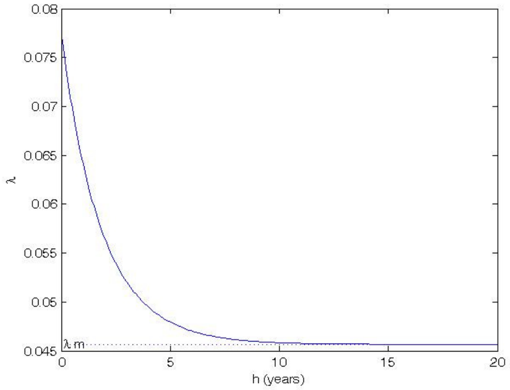

Figure 1. $\delta > 0$ and $\alpha > \Gamma$. In this case λ is globally decreasing and convex in h, tending asymptotically to λ_m as $h \to \infty$.

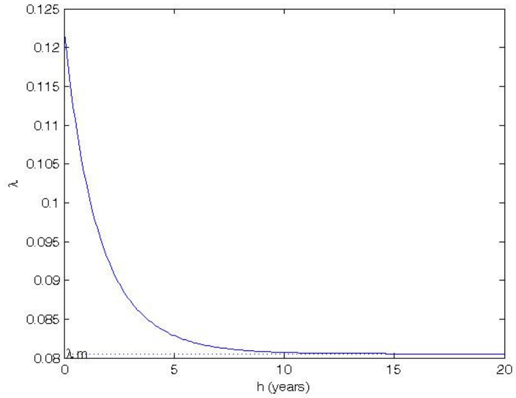

Figure 2. $\delta > 0$ and $\Gamma > \alpha > r$. Now, the maximum value for λ occurs at $\hat{h} < 0$, to the left of the origin and $\partial\lambda(\alpha,0)/\partial h < 0$. Therefore λ is again globally decreasing in h, for positive values of h. λ may initially be convex or concave for positive h, depending upon whether $\bar{h}$ is negative or positive.

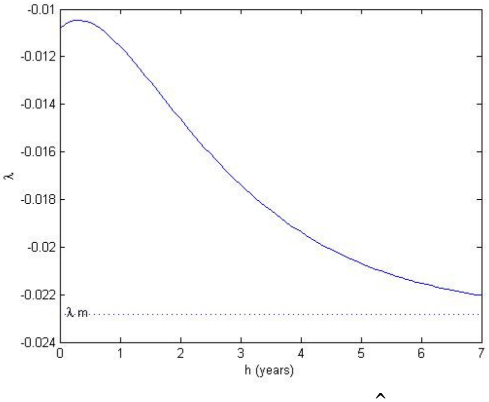

Figure 3. $\delta > 0$ and $r > \alpha$. In this case the maximum point occurs at $\hat{h}$, to the right of the origin ($\hat{h}$ = 0.3015). It can be shown that $\lambda(\alpha,0)$ may be greater or less than λ_m. In the former case, λ starts above λ_m, rises to a maximum (it is hard to see because $\hat{h}$ is very small), and then falls asymptotically to λ_m in the latter case, λ actually starts below λ_m, climbs above it, reaches a maximum and then falls.

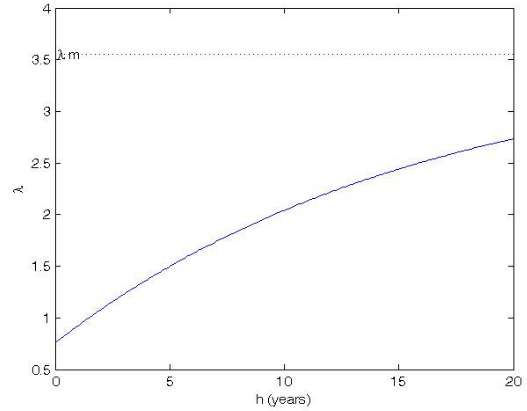

Figure 4. $\delta < 0$ and $\alpha > \Gamma$. In this case λ is globally increasing and concave in h, tending asymptotically to λ_m as $h \to \infty$.

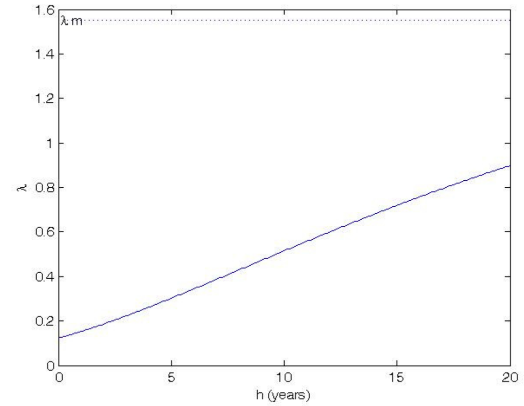

Figure 5. $\delta < 0$ and $\Gamma > \alpha > r$. λ is again globally increasing in h, for positive values of h. λ may initially be convex or concave for positive h. This graph shows the case where λ is initially convex.

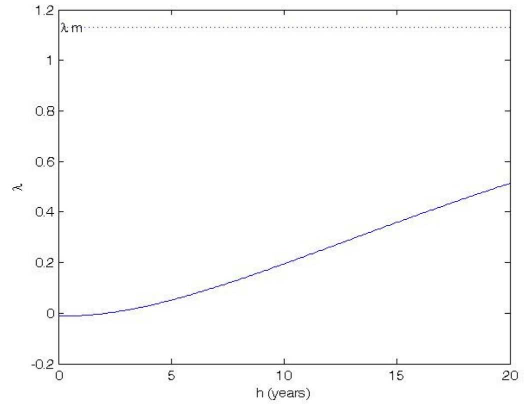

Figure 6. $\delta < 0$ and $r > \alpha$. In this case $\hat{h}$ occurs to the right of the origin (it is hard to see in the graph because $\hat{h}$ =0.323 is very small), λ initially decreases in h, reaches a minimum and then increases asymptotically to λ_m as $h \to \infty$.